# A parallel friendly non local OSM for electromagnetism

Phd student: Roxane Delville-Atchekzai [1,2]
Director: Xavier Claeys [2]
Supervisor: Matthieu Lecouvez [1]

[1]CEA-Cesta, [2]LJLL Sorbonne Université

May 15, 2023

# Numerical problem under study

## Harmonic Maxwell problem

$\Omega$ polyhedral domain with border $\partial\Omega$

$$\left\{\begin{array}{lcll} \mathbf{E} \in H_{curl}(\Omega) & & & \\ \nabla \times \nabla \times \mathbf{E} - \omega^2 \mathbf{E} & = & 0 & \Omega \\ \mathbf{n} \times \nabla \times \mathbf{E} - i\omega \mathbf{n} \times \mathbf{E} \times \mathbf{n} & = & \mathbf{g} & \partial\Omega \end{array}\right. \quad (1)$$

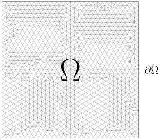
$H_{curl}(\Omega) := \{u \in (L^2(\partial\Omega))^3 : \nabla \times u \in (L^2(\partial\Omega))^3\}$
$\mathbf{g} \in (L^2(\partial\Omega))^3$

Equivalent variational formulation

$$\left\{\begin{array}{l} \text{Find } \mathbf{u} \in H_{curl}(\Omega) : \\ a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \qquad \forall \mathbf{v} \in H_{curl}(\Omega) \\ a(\mathbf{u}, \mathbf{v}) := \int_\Omega \nabla \times \mathbf{u} \cdot \overline{\nabla \times \mathbf{v}} - \omega^2 \epsilon \mathbf{u} \cdot \bar{\mathbf{v}} dx \\ -i \int_{\partial\Omega} \omega (\mathbf{u} \times \mathbf{n}) \cdot (\bar{\mathbf{v}} \times \mathbf{n}) d\sigma \\ l(\mathbf{v}) := i\omega \int_{\partial\Omega} \mathbf{n} \times \mathbf{g} \cdot \mathbf{n} \times \bar{\mathbf{v}} dx \end{array}\right. \quad (2)$$

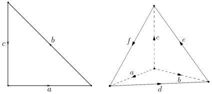
## Discretization with Nédélec finite elements

Simplicial triangulation $\mathcal{T}_h$, we assume $\overline{\Omega} = \bigcup_{t_h \in \mathcal{T}_h} t_h$




$$\left\{\begin{array}{l} \text{Find } u_h \in V_h(\Omega) \subseteq H_{curl}(\Omega) \\ \mathbf{A}(u_h, v) = l(v_h) \quad \forall v_h \in V_h(\Omega) \end{array}\right.$$

$$u_h = \sum_{i=1}^{N} \theta_i \varphi_i \qquad \theta_i = \int_{e_i} u_h(l) \cdot \tau_{e_i} dl$$



# Outline

1. What is non local OSM ?
   - Optimized Schwarz Method
   - Non local transmission
2. Make non local OSM scalable
   - Numerical solve of skeleton problem
   - Nodal auxiliary space preconditioning

# Outline

1. What is non local OSM ?
   - Optimized Schwarz Method
   - Non local transmission
2. Make non local OSM scalable
   - Numerical solve of skeleton problem
   - Nodal auxiliary space preconditioning

# Domain decomposition setting

Non-overlapping domain decomposition is called Optimized Schwarz Method (OSM).

Partition of Ω into J non-overlapping subdomains

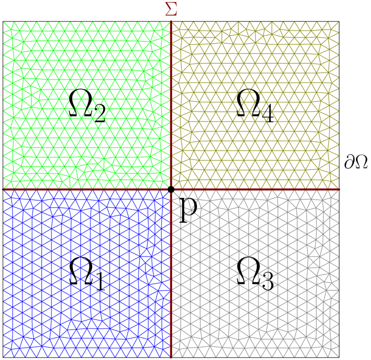
$$\overline{\Omega} = \bigcup_{j=1}^{J} \overline{\Omega_j} \quad \Omega_j \cap \Omega_k = \emptyset \quad \forall j \neq k$$

Skeleton Σ is the union of subdomain borders or interfaces

$$\Sigma := \bigcup_{j=1}^{J} \overline{\Gamma_j} \quad \Gamma_j = \partial\Omega_j \text{ or } \Gamma_j = \partial\Omega_j \backslash (\partial\Omega_j \cap \partial\Omega)$$

Subdomain interfaces

$$\Gamma_{jk} = \partial\Omega_j \cap \partial\Omega_k = \Gamma_j \cap \Gamma_k \quad \forall j \neq k$$



# Crosspoint issue in classical OSM

Because of convergence issues for time-harmonic wave problems, we need a special choice of Robin condition at subdomain interfaces.

[Després 1991][Dolean, Gander,Gerardo,Giorda,2009]

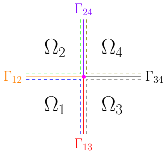
Local subproblems on $\Omega_j$ with local transmission conditions with Dirichlet-to-Neumann operator $\mathcal{DTN}$.

$$\begin{cases} \text{Find } \mathbf{E_j} \in H_{curl}(\Omega_j) & & \\ \nabla \times \nabla \times \mathbf{E_j} - \omega^2 \mathbf{E_j} & = & 0 \quad \Omega_j \\ \mathbf{n}_j \times \nabla \times \mathbf{E_j} - i\omega \mathbf{n} \times \mathbf{E_j} \times \mathbf{n_j} & = & \mathbf{g} \quad \partial\Omega_j \cap \partial\Omega \\ \mathbf{n}_j \times \mathbf{E_j} + \mathcal{DTN}(\mathbf{E_j}) & = & \mathbf{n}_k \times \mathbf{E_k} + \mathcal{DTN}(\mathbf{E_k}) \quad \Gamma_{jk} \end{cases}$$

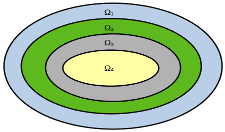
Reconstruction of global solution **E** with partition of unity $\chi$

$$\mathbf{E} = \sum_{j=1}^{J} \chi_j \mathbf{E}_j$$

Defining local transmission conditions and guarantee robust convergence in general setting is difficult. [Gander,Kwok,2012]




Avoiding crosspoint in domain partition is very restrictive and an obstacle for scalability.



# Function spaces for non local OSM

This part follows [Claeys, Parolin 2022].

- Reformulate sequential problem into a skeleton problem on $\mathbb{V}_h(\Sigma)$
- Impose Dirichlet continuity conditions across subdomain interfaces non locally

## Sequential problem

$$\begin{cases} \text{Find } u \in V_h(\Omega) \\ \mathbf{A}(u,v) = l(v) \quad \forall v \in V_h(\Omega) \end{cases} \tag{3}$$

## Multi-trace $\mathbb{V}_h(\Sigma)$ and single trace $\mathbb{X}_h(\Sigma)$ spaces

$$\begin{aligned} \mathbb{V}_h(\Omega) &:= V_h(\Omega_1) \times \cdots \times V_h(\Omega_J) \\ &= \{u \in L^2(\Omega) : u_{|\Omega_j} \in V_h(\Omega_j) \quad \forall 1 \le j \le J\} \\ \mathbb{V}_h(\Sigma) &:= V_h(\Gamma_1) \times \cdots \times V_h(\Gamma_J) = \{u_{|\Sigma} : u \in \mathbb{V}_h(\Omega)\} \\ \mathbb{X}_h(\Sigma) &:= \{(u_{|\Gamma_j})_{1 \le j \le J} \in \mathbb{V}_h(\Sigma) : u \in V_h(\Omega)\} \subset \mathbb{V}_h(\Sigma) \end{aligned} \tag{4}$$

# Decoupling the subdomains

**Step 1:** Work in $\mathbb{V}_h(\Omega)$ + impose continuity condition: $u_{|_\Sigma} \in \mathbb{X}_h(\Sigma)$

Formulation with Lagrange multiplier $q \in \mathbb{X}_h(\Sigma)^\perp$

$$\left\{ \begin{array}{lcl} \text{Find } (u,q) \in \mathbb{V}_h(\Omega) \times \mathbb{X}_h(\Sigma)^\perp & & \\ \mathbf{A}(u,v) - \mathbf{T}(q, v_{|_\Sigma}) & = & l(v) \quad \forall v \in \mathbb{V}_h(\Omega) \\ \mathbf{T}(u_{|_\Sigma}, w) & = & 0 \quad \forall w \in \mathbb{X}_h(\Sigma)^\perp \end{array} \right. \tag{5}$$

Need of a characterization of $\mathbb{X}_h(\Sigma)$ without $V_h(\Omega)$: work in $\mathbb{V}_h(\Sigma)$ in second equation of (5) $\Rightarrow$ use a trace exchange operator for continuity condition.

# The exchange operator

**Scalar product on multi-trace space $(\mathbb{V}_h(\Sigma), \|\cdot\|_{\mathbf{T}})$**

$\mathbf{T} : \mathbb{V}_h(\Sigma) \times \mathbb{V}_h(\Sigma) \to \mathbb{C}$ referred as the impedance operator.

**Definition of $\mathbf{\Pi}$**

Operator $\mathbf{\Pi} : \mathbb{V}_h(\Sigma) \to \mathbb{V}_h(\Sigma)$ is such that $\frac{\mathbf{Id}+\mathbf{\Pi}}{2}$ is $\mathbf{P}_{\mathbf{T}}$, the $\mathbf{T}$-orthogonal projector onto the single-trace space $\mathbb{X}_h(\Sigma)$.

**Computing $\mathbf{\Pi}$**

$$\begin{gathered}\mathbf{\Pi}(q) := -q + 2p \text{ where } p \in \mathbb{V}_h(\Sigma) \text{ solves} \\ \mathbf{T}(p, w) = \mathbf{T}(q, w) \quad \forall w \in \mathbb{X}_h(\Sigma)\end{gathered} \tag{6}$$

# Rewrite problem with exchange operator

**Step 2:** Inject the characterization of $\mathbb{X}_h(\Sigma)$ into the problem.

## Characterization of $\mathbb{X}_h(\Sigma)$ using $\mathbf{\Pi}$

For any pair $(w, q) \in \mathbb{V}_h(\Sigma) \times \mathbb{V}_h(\Sigma)$, the following equivalence holds:

$$(w, q) \in \mathbb{X}_h(\Sigma) \times \mathbb{X}_h(\Sigma)^{\perp} \iff -q + iw = \mathbf{\Pi}(q + iw) \tag{7}$$

## Decoupled system

$$\begin{cases} \text{Find } (u, q) \in \mathbb{V}_h(\Omega) \times \mathbb{V}_h(\Sigma) \\ \mathbf{A}(u, v) - \mathbf{T}(q, v|_\Sigma) = l(v) \quad \forall v \in \mathbb{V}_h(\Omega) \\ q - iu|_\Sigma = -\mathbf{\Pi}(q + iu|_\Sigma) \end{cases} \tag{8}$$

# The scattering operator

**Step 3:** Eliminate volume unknowns in (9) to reduce the problem to the skeleton $\Sigma$. $\Rightarrow$ need of an operator $\mathbf{S} : p \mapsto \tilde{u}|_\Sigma$ with $u = \tilde{u} + u_*$.

Decoupled system after change of variable $p = q - iu|_\Sigma$

$$\left\{\begin{array}{lcl} \text{Find } (u, p) \in \mathbb{V}_h(\Omega) \times \mathbb{V}_h(\Sigma) & & \\ \mathbf{A}(u, v) - i\mathbf{T}(u|_\Sigma, v|_\Sigma) & = & \mathbf{T}(p, v|_\Sigma) + l(v) \quad \forall v \in \mathbb{V}_h(\Omega) \\ p & = & -\mathbf{\Pi}(p + 2iu|_\Sigma) \end{array}\right. \tag{9}$$

Definition of scattering operator

Scattering operator $\mathbf{S} : \mathbb{V}_h(\Sigma) \to \mathbb{V}_h(\Sigma)$

$$\left\{\begin{array}{lcl} \mathbf{S}(p) & = & p + 2i\tilde{u}|_\Sigma \text{ where } \tilde{u} \text{ solves:} \\ \mathbf{A}(\tilde{u}, v) - i\mathbf{T}(\tilde{u}|_\Sigma, v|_\Sigma) & = & \mathbf{T}(p, v|_\Sigma) \quad \forall v \in \mathbb{V}_h(\Omega) \end{array}\right. \tag{10}$$

# Final form of the problem

**Step 4:** Inject the scattering operator in the decoupled system.

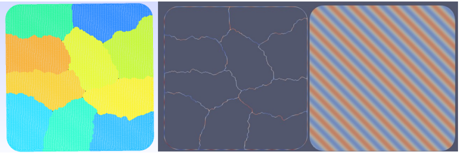
The skeleton problem

$$\begin{cases} \text{Find } p \in \mathbb{V}_h(\Sigma) \\ (\mathbf{Id} + \mathbf{\Pi}\mathbf{S})p = f \end{cases} \quad (11)$$

where $f = -2i\mathbf{\Pi}(u_*|_\Sigma)$ where $u_* \in \mathbb{V}_h(\Omega)$ solves

$$\mathbf{A}(u_*, v) - i\mathbf{T}(u_*|_\Sigma, v|_\Sigma) = l(v) \quad \forall v \in \mathbb{V}_h(\Omega)$$

**Step 5:** After solving the skeleton problem solution $p \in \mathbb{V}_h(\Sigma)$, recover global volume solution $u \in \mathbb{V}_h(\Omega)$



# Outline

1. What is non local OSM ?
   - Optimized Schwarz Method
   - Non local transmission
2. Make non local OSM scalable
   - Numerical solve of skeleton problem
   - Nodal auxiliary space preconditioning

# Two nested iterative methods: Richardson and Conjugate Gradient

The skeleton problem is solved with Richardson algorithm, the speed of convergence is geometric depending on strong coercivity constant $\gamma_h$.

[Claeys,Parolin 2022]

$$\begin{cases} \text{Find } p \in \mathbb{V}_h(\Sigma) \\ (\mathbf{Id} + \mathbf{\Pi}\mathbf{S})p = f \end{cases}$$

**Algorithm** Richardson-DDM

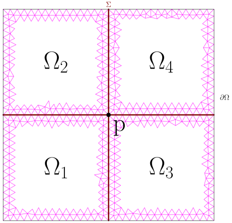
$n \leftarrow 0$
$p_n \leftarrow (0, \dots, 0)$
$v \leftarrow f_{in}$
**while** $\|v\|_{\mathbf{T}} \geq \nu$ **do**
  $p_n \leftarrow p_n + \alpha v$
  $v \leftarrow f_{in} - (\mathbf{Id} + \mathbf{\Pi}\mathbf{S})p_n$
  $n \leftarrow n + 1$
**end while**
**return** $p_n$

- Skeleton problem solved with iterative algorithm like Richardson or GMRES
- Conjugate Gradient to compute $\mathbf{\Pi}$ at each iteration of Richardson-DDM

# Schur complement based impedance

Consider the following positive Maxwell problem on the thick skeleton

$$
\begin{cases}
\text{Find } \phi \in H_{curl}(\Omega'_j) \\
\nabla \times \nabla \times \phi + \phi = 0 & \Omega'_j \\
\phi = v & \Gamma_j \\
n \times \nabla \times \phi + \phi = 0 & \partial\Omega'_j \setminus (\partial\Omega'_j \cap \Gamma_j)
\end{cases} \tag{12}
$$




Matrix form:

$$
\begin{pmatrix} C & -B^* \\ B & 0 \end{pmatrix} \begin{pmatrix} \phi \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{13}
$$

## Definition of Schur complement based impedance

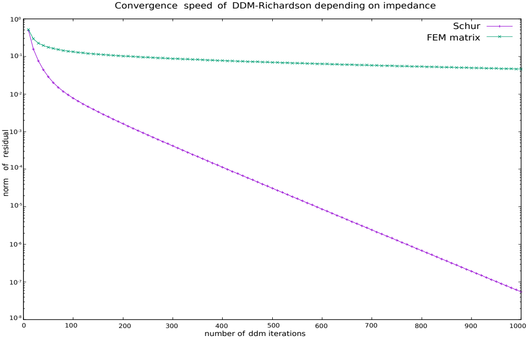
$$
\begin{aligned}
\mathbf{T} : H^{-1/2}(\Gamma_j) &\rightarrow H^{1/2}(\Gamma_j) \\
v &\mapsto q
\end{aligned} \tag{14}
$$

# Schur complement based impedance: convergence speed

## Convergence speed for Schur complement based impedance [Claeys,Parolin 2022]

Unconditionnal h-uniform stability for Schur complement based impedance

$$\frac{\|p_n - p\|_{\mathbf{T}}}{\|p_0 - p\|_{\mathbf{T}}} \leq (1 - \alpha(1-\alpha)\gamma_*^2)^{\frac{n}{2}} \quad \forall h \in (0, h_*) \quad \forall n \geq 0 \qquad (15)$$



# Projection problem

Computing the exchange $\mathbf{\Pi}$ implies solving the following projection problem.
Restriction operator $\mathbf{R} : V_h(\Sigma) \to \mathbb{X}_h(\Sigma) \subset \mathbb{V}_h(\Sigma)$
$\mathbf{R}(q) = (\mathbf{R}_1(q), \dots, \mathbf{R}_J(q)) = (q|_{\Gamma_1}, \dots, q|_{\Gamma_J})$

**Projection problem**

$$\forall x \in \mathbb{V}_h(\Sigma) \qquad \mathbf{P}(x) = \mathbf{R}y \quad \text{where } y \text{ solves } \mathbf{R}^T\mathbf{T}\mathbf{R}y = \mathbf{R}^T\mathbf{T}x \tag{16}$$

Symmetric positive definite problem solved with conjugate gradient (CG).

**Convergence speed of CG**

$$\|e_k\| \leq \|e_0\| \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \tag{17}$$

# Preconditioning positive Maxwell problem

The nodal auxiliary space preconditioning technique for positive Maxwell problems is promising. [Hiptmair,Xu,2007]

The main difficulty for Maxwell problems: infinite dimensional kernel of curl operator in $H_{curl}(\Omega')$ is hard to capture with a classical preconditioner.

$$H_{curl}(\Omega') = (H^1(\Omega'))^3 \oplus ker(curl)$$

## Fictitious space preconditioner

$(V, \|.\|_A)$ Hilbert space.

- Fictitious space $(\overline{V}, \|.\|_{\overline{A}})$
- Linear, continuous, surjective transfer operator $\mathcal{T} : \overline{V} \to V$

Preconditioner $M := \mathcal{T} \circ \overline{A}^{-1} \circ \mathcal{T}^*$
Fictitious space lemma: $\kappa(MA) \leq (c_0 c_1)^2$ where $c_0$ and $c_1$ depend on $V, \overline{V}, \mathcal{T}$

# Nodal auxiliary space preconditioner

How to build fictitious space $\overline{V}$ and transfer operator $\mathcal{T}$ ?
$\Rightarrow$ with nodal auxiliary spaces $(W_1, \ldots, W_m)$ determined with stable regular decomposition of $H_{curl}(\Omega')$

## Fictitious space

Fictitious space $\overline{V} = V \times W_1 \times W_2$ with $W_1 = (H^1(\Omega'))^3$ and $W_2 = H_{grad}(\Omega)$

For discretization, replace with corresponding conforming finite element spaces: $(\mathbb{P}^1_{lag}(\Omega'))^3$ and $V_h(grad, \Omega')$
Corresponding transfer operators $P_{curl} : (\mathbb{P}^1_{lag}(\Omega'))^3 \to V_h(\Omega', curl)$,
$G : V_h(grad, \Omega') \to V_h(\Omega', curl)$

## Formula of preconditioner

$$M_{curl} = D_A^{-1} + P_{curl}(L + \tau M)^{-1} P_{curl}^T + \tau^{-1} G(-\Delta)^{-1} G^T \qquad (18)$$

# Conclusion and prospects

We have explored two promising strategies to tackle the non-locality of the exchange operator: preconditionning and combining recycling and truncation of conjugate gradient.

Future works:

- Design CG recycling strategy for GMRES DDM
- Analysis of recycling and truncation of CG
- Test numerical efficiency of nodal auxiliary space preconditionner
- Parallelize the code of the non local OSM

# References


- Claeys, Xavier (2020). "Non-local variant of the optimised Schwarz method for arbitrary non-overlapping subdomain partitions". In: *ESAIM: Mathematical Modelling and Numerical Analysis* 55.2, pp. 429–448.
- Claeys, Xavier, Francis Collino, and Emile Parolin (Aug. 2021). "Matrix form of nonlocal OSM for electromagnetics". In: *arXiv:2108.11352 [cs, math]*.
- Claeys, Xavier and Emile Parolin (June 1, 2022). "Robust treatment of cross-points in optimized Schwarz methods". In: *Numerische Mathematik* 151.2, pp. 405–442. ISSN: 0945-3245.
- Després, B. (1991). "Méthodes de décomposition de domaine pour la propagation d'ondes en régime harmonique.Le théorème de Borg pour l'équation de Hill vectorielle".
- Dolean, V., M.J. Gander, and L. Gerardo-Giorda (Jan. 2009). "Optimized Schwarz Methods for Maxwell's Equations". In: *SIAM Journal on Scientific Computing* 31.3, pp. 2193–2213.
- Gander, Martin J. and Felix Kwok (Jan. 2012). "Best Robin Parameters for Optimized Schwarz Methods at Cross Points". In: *SIAM Journal on Scientific Computing* 34.4, A1849–A1879.
- Hiptmair, Ralf and Jinchao Xu (July 1, 2008). "Auxiliary Space Preconditioning for Edge Elements". In: *Magnetics, IEEE Transactions on* 44, pp. 938–941. DOI: 10.1109/TMAG.2007.916508.

Thank you for your attention