Trefftz-like Coarse Space for Perforated Domains

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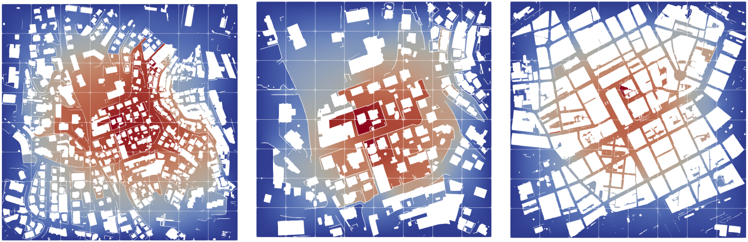
Outline

Motivation

- Efficiently solve problems on perforated domains.
  - Numerous holes representing buildings and walls in urban data;
  - Can be considered a heterogeneous domain with coefficients $0, 1$.
  - Expect corner singularities
  - Want to avoid global fine-scale solve.
- We begin with the linear Poisson equation before moving to nonlinear problems (Diffusive Wave model).
- Applications: flood modelling in urban areas.



Model PDE: Linear

- D: Open simply connected polygonal domain in $\mathbb{R}^2$;
- $(\Omega_{S,k})_k$: Finite family of perforations in D;
- $\Omega_S = \bigcup_k \Omega_{S,k}$ and $\Omega = D \setminus \overline{\Omega_S}$.

$$\begin{cases} -\Delta u & = & f & \text{in} & \Omega, \\ \dfrac{\partial u}{\partial n} & = & 0 & \text{on} & \partial\Omega \cap \partial\Omega_S, \\ u & = & 0 & \text{on} & \partial\Omega \setminus \partial\Omega_S. \end{cases}$$

With a P1 finite element discretization, this discretely becomes the linear system

$$\mathbf{Au} = \mathbf{f}.$$

Domain Decomposition Approach

- 'Divide and conquer': Break up problem into subdomains;
- Two levels of discretization: 'Coarse' and 'fine';
- Local subdomain solves can be done in parallel;
- Can use overlapping Schwarz methods as iterative solver or as preconditioner for Krylov;

Idea: Solve model problem on each subdomain locally, with boundary conditions taken from adjacent subdomains when possible.

Parallel Schwarz Introduction for $\mathcal{L}u = f$: 2 subdomains

Continuously, the local classical additive Schwarz iteration is given by

$$\begin{aligned} \mathcal{L}u_1^{n+1} &= f \quad \text{in} \quad \Omega_1 & \qquad \mathcal{L}u_2^{n+1} &= f \quad \text{in} \quad \Omega_2 \\ u_1^{n+1} &= u_2^n \quad \text{on} \quad \partial\Omega_1 \cap \Omega_2 & \qquad u_2^{n+1} &= u_1^n \quad \text{on} \quad \partial\Omega_2 \cap \Omega_1 \end{aligned}$$

Algebraically, the global stationary (RAS) iteration becomes

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \left(\sum_{j=1}^{2} \mathbf{R}_j^T \mathbf{D}_j (\mathbf{R}_j \mathbf{A} \mathbf{R}_j^T)^{-1} \mathbf{R}_j \right) (\mathbf{f} - \mathbf{A}\mathbf{u}^n)$$

and the preconditioned system is given by

$$\left(\sum_{j=1}^{2} \mathbf{R}_j^T \mathbf{D}_j (\mathbf{R}_j \mathbf{A} \mathbf{R}_j^T)^{-1} \mathbf{R}_j \right) \mathbf{A}\mathbf{u} = \left(\sum_{j=1}^{2} \mathbf{R}_j^T \mathbf{D}_j (\mathbf{R}_j \mathbf{A} \mathbf{R}_j^T)^{-1} \mathbf{R}_j \right) \mathbf{f}$$

- $\mathbf{R}_j$ notation allows for global iteration, algebraic definition, overlapping subdomains.

1D example- Restriction, POU matrices

Given set of indices $\mathcal{N} = \{0,1,2,3,4\}$: partitioned into $\mathcal{N}_1 = \{0,1,2,3\}$ and $\mathcal{N}_2 = \{2,3,4\}$, restriction and partition of unity matrices are given as

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{R}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{D}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad \mathbf{D}_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Satisfies $\mathbf{I} = \sum_{j=1}^{2} \mathbf{R}_j^T \mathbf{D}_j \mathbf{R}_j$.

Need for coarse correction

- Coarse corrections allows for global communication between all subdomains.
- Coarse correction (two-level methods) necessary for scalability for large number of subdomains.
- Generally, without coarse correction: Iterations scale with N.

2-level RAS iteration: N Subdomains

Combine (multiplicitavely) the 1-level RAS iteration

$$M_{RAS,1}^{-1} = \sum_{j=1}^{N} \mathbf{R}_j^T \mathbf{D}_j (\mathbf{R}_j \mathbf{A} \mathbf{R}_j^T)^{-1} \mathbf{R}_j$$

with the coarse approximation

$$M_0^{-1} = \mathbf{R}_0^T (\mathbf{R}_0 \mathbf{A} \mathbf{R}_0^T)^{-1} \mathbf{R}_0.$$

and solve

$$\mathbf{u}^{n+\frac{1}{2}} = \mathbf{u}^n + M_{RAS,1}^{-1}(\mathbf{f} - \mathbf{A}\mathbf{u}^n),$$
$$\mathbf{u}^{n+1} = \mathbf{u}^{n+\frac{1}{2}} + M_0^{-1}(\mathbf{f} - \mathbf{A}\mathbf{u}^{n+\frac{1}{2}}),$$

- $\mathbf{R}_j$: Correspond to overlapping subdomains.

The 2-level preconditioner for Krylov

Combine (additively) the 1-level RAS iteration

$$M_{RAS,1}^{-1} = \sum_{j=1}^{N} \mathbf{R}_j^T \mathbf{D}_j (\mathbf{R}_j \mathbf{A} \mathbf{R}_j^T)^{-1} \mathbf{R}_j$$

with the coarse approximation

$$M_0^{-1} = \mathbf{R}_0^T (\mathbf{R}_0 \mathbf{A} \mathbf{R}_0^T)^{-1} \mathbf{R}_0.$$

to give

$$M_{RAS,2}^{-1} = M_0^{-1} + M_{RAS,1}^{-1}.$$

and solve

$$M_{RAS,2}^{-1} \mathbf{A} \mathbf{u} = M_{RAS2}^{-1} \mathbf{f}.$$

(Some) existing overlapping Schwarz coarse spaces

- Nicolaides: Piecewise constant by subdomain;
- Spectral spaces (eigenvalue problems): DtN, GenEO, SHEM (spectrally enriched MSFEM);
- Energy-minimizing spaces: GDSW, AGDSW, RGDSW;
- Multi-scale FEM: MsFEM
  - Numerically compute harmonic basis functions.
  - Used to approximate solution on coarse grid, but can use as DD coarse space!

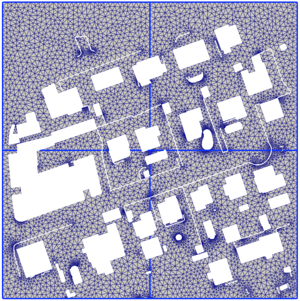
Choice of coarse space

- Idea: want to take advantage of a-priori location of perforations (buildings/walls);
- Want robustness with respect to perforation size/location (even along subdomain interfaces);
- Want to choose a coarse space with approximation properties to improve convergence;
- Choose: Local harmonic basis functions occuring at intersection of a perforation with the coarse skeleton.
  - Think of as 'enriching' MsFEM coarse space.
  - Works on nonoverlapping subdomains Ω_j'.

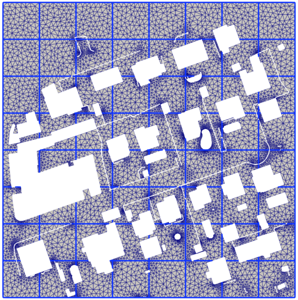
Coarse-cell conforming triangulation

Mesh generation process:

- Larger $N \rightarrowtail$ more basis functions, larger coarse matrix ;
- Triangulate after nonoverlapping coarse cell partitioning Ω'_j;
- Overlap subdomains by layers of triangles for RAS.



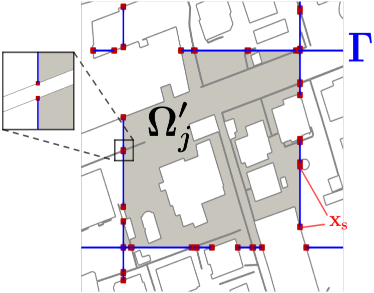
2×2 subdomains



8×8 subdomains

Coarse grid nodes for coarse space basis functions

- Nonoverlapping skeleton: $\Gamma = \bigcup_{j \in \{1,\ldots,N\}} \partial\Omega_j'$;
- $(e_k)_{k=1,\ldots,N_e}$: Partitioning of Γ;
  - each "coarse edge" e_k is an open planar segment;
- Set of coarse grid nodes: $\bigcup_{k=1,\ldots,N_e} \partial e_k$
- $(\phi_s)_{s \in \{1,\ldots,N_x\}}$: Locally harmonic basis functions for each coarse grid node.
- # of coarse grid nodes is **automatically** generated.

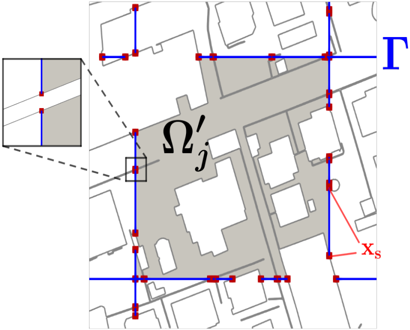


Basis functions: boundary conditions

For each coarse grid node $\mathbf{x}_s$, define $g_s : \Gamma \to [0,1]$ as: for $i = 1, \ldots N_{\mathbf{x}}$,

$$g_s(\mathbf{x}_i) = \begin{cases} 1, & s = i, \\ 0, & s \neq i, \end{cases}$$

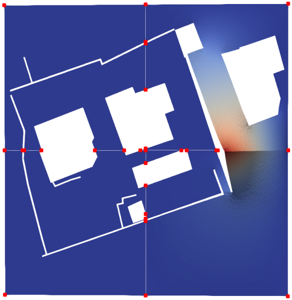
- g_s is linearly extended on the remainder of Γ.
- Can also include higher-order polynomials on coarse edges.



Basis functions: Harmonic local solutions

For all nonoverlapping $(\Omega'_j)_{j\in\{1,\ldots,N\}}$ and $s = 1,\ldots,N_{\mathbf{x}}$, to obtain $\phi_{s,j} = \phi_s|_{\Omega_j}$, solve

$$\begin{cases} -\Delta\phi_{s,j} & = & 0 & \text{in} & \Omega'_j, \\ -\dfrac{\partial\phi_{s,j}}{\partial n} & = & 0 & \text{on} & \partial\Omega'_j \cap \partial\Omega_S, \\ \phi_{s,j} & = & g_s & \text{on} & \partial\Omega'_j \setminus \partial\Omega_S. \end{cases}$$



- $\text{supp}(\phi_s) = \{\bigcup_j \Omega'_j \mid \mathbf{x}_s \text{ is a coarse grid node belonging to } \partial\Omega'_j\}$.
- Continuously, the coarse space is given by

$$V_H = \text{span}\{\phi_s\}.$$

Approximation properties: Coarse approximation

Discretely, given

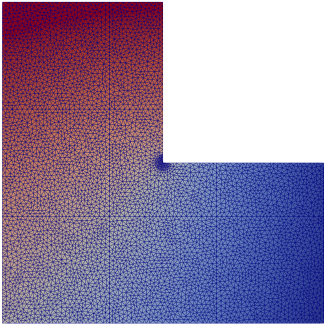
$$M_0^{-1} = \mathbf{R}_0^T(\mathbf{R}_0\mathbf{A}\mathbf{R}_0^T)^{-1}\mathbf{R}_0.$$

the coarse approximation is the solution of

$$\mathbf{u}_H = M_0^{-1}\mathbf{f}.$$

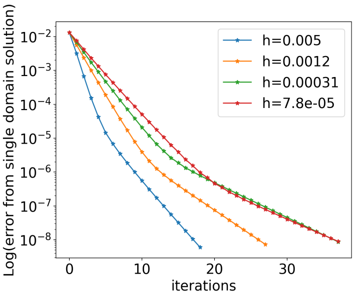
- Can use u_H as initial iterate for iteration, Krylov methods.

Experiment 1: Iterative RAS, L-shaped domain

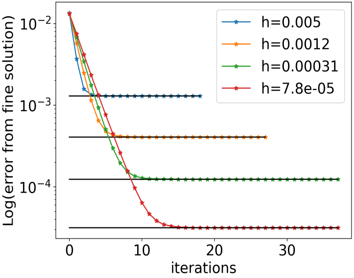


- Provide iterative RAS results for preliminary L-shaped domain;
    - L-shaped domain: Square domain with one perforation;
    - Allows us to compare to analytical solution.
    - Perform additional refinement at the singularity to improve convergence and FE error;
- Keep N constant, vary h and improve FE error;
- In spirit of iterative methods.

Numerical Results: Iterative RAS (L-shaped domain)



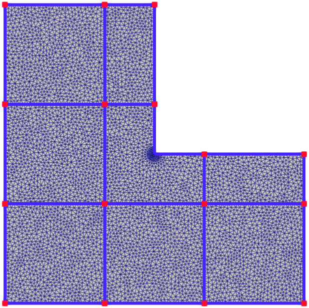
SD error



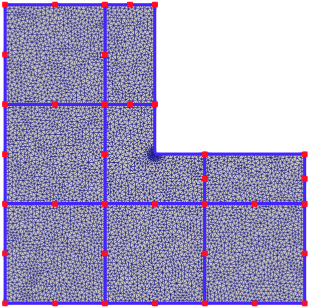
True error

- SD error: Error from algebraic single domain FE solution;
- True error: Error from analytical true solution.

Edge refinement



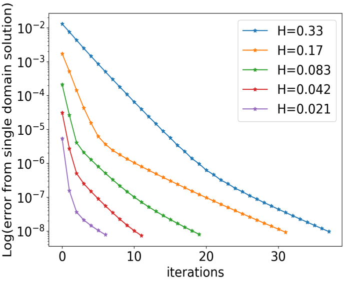
Orig. coarse grid nodes



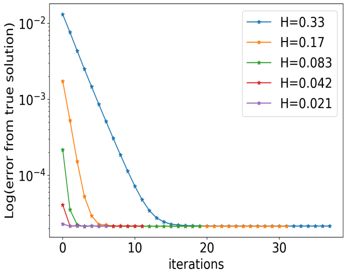
Additional edge refinement

- Improves coarse approximation;
- No changes to coarse skeleton Γ.
- Idea from MHM literature.

Numerical Results: Iterative RAS (L-shaped domain) Edge refinement



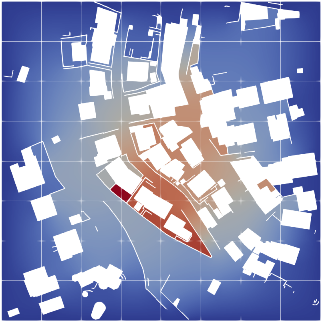
SD error



True error

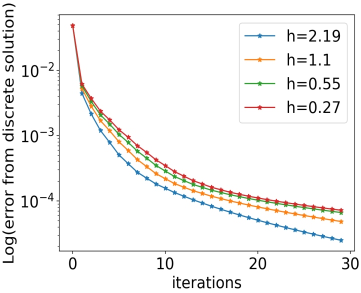
- Vary $H = max_{k=1,\ldots,N_e}|e_k|$, keep h constant;
- Edge refinement provides additional acceleration (better coarse approx., steeper slope).

Experiment 2: Iterative+Krylov, real data set

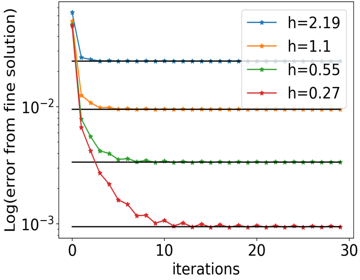


- Provide same iterative convergence curves as L-shaped domain;
- Also provide convergence curves for preconditioned GMRES;
- Multiple singularities and no analytical solution available.

Numerical Results: Iterative RAS (Real data)



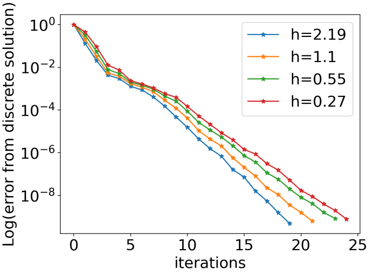
SD error



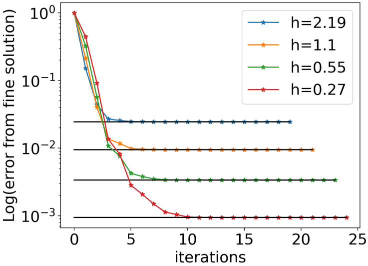
True error

- SD error: Error from algebraic single domain FE solution;
- True error: Error from fine FE solution.

Numerical Results: Krylov

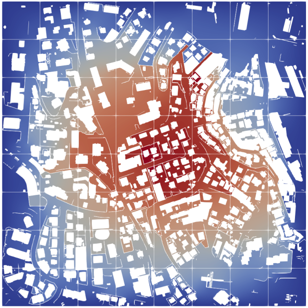


SD error



True error

Experiment 3: Krylov Scalability, large real data set



$\approx$ 300K DOFS in FE triangulation.

- Want to show scalability:
- "Strong" scalability tests: Keep model domain and h constant, vary N.

Numerical Results: Krylov (table)

	Trefftz		
	it.		dim. (rel)
N	min	$\frac{H}{20}$	
16	56	22	400 (16.0)
64	56	26	880 (10.9)
256	59	30	1912 (6.6)
1024	61	28	4253 (3.9)

- Relative dimension (rel): Compared to would-be homogeneous domain, $\frac{\dim(R_0)}{(\sqrt{N}+1)^2}$.
- Relative dimension reduces as N increases;
- Trefftz-like space produces scalable, accelerated iterations.

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- Achieve fine-scale error in a small number of iterations, limited by finite element error;
- Krylov: Trefftz is **Robust** with respect to number of subdomains on a fixed total domain size, and provides an additional **acceleration** in terms of Krylov iteration count.
- However, the dimension of the Trefftz-like coarse space is large and controlled by the model geometry.

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Thank you for your time!