Trefftz-like Coarse Space for Perforated Domains

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IMPDE 2023, May 15th, 2023
Outline

Model problem and Introduction

Overlapping Schwarz methods

Construction of Coarse Space

Numerical Results: Iterative

Numerical Results: Krylov

Closing Remarks
Motivation

- Efficiently solve problems on perforated domains.
  - Numerous holes representing buildings and walls in urban data;
  - Can be considered a heterogeneous domain with coefficients $0, 1$.
  - Expect corner singularities
  - Want to avoid global fine-scale solve.

- We begin with the linear Poisson equation before moving to nonlinear problems (Diffusive Wave model).

- Applications: flood modelling in urban areas.
Model PDE: Linear

- $D$: Open simply connected polygonal domain in $\mathbb{R}^2$;
- $(\Omega_{S,k})_k$: Finite family of perforations in $D$;
- $\Omega_S = \bigcup_k \Omega_{S,k}$ and $\Omega = D \setminus \overline{\Omega_S}$.

\[
\begin{cases}
-\Delta u = f \quad \text{in} \quad \Omega, \\
\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega \cap \partial \Omega_S, \\
u = 0 \quad \text{on} \quad \partial \Omega \setminus \partial \Omega_S.
\end{cases}
\]

With a P1 finite element discretization, this discretely becomes the linear system

\[Au = f.\]
Domain Decomposition Approach

- 'Divide and conquer': Break up problem into subdomains;
- Two levels of discretization: 'Coarse' and 'fine';
- Local subdomain solves can be done in parallel;
- Can use overlapping Schwarz methods as iterative solver or as preconditioner for Krylov;

Idea: Solve model problem on each subdomain locally, with boundary conditions taken from adjacent subdomains when possible.
Parallel Schwarz Introduction for $\mathcal{L}u = f$: 2 subdomains

Continuously, the local classical additive Schwarz iteration is given by

$$
\mathcal{L}u^{n+1}_1 = f \quad \text{in} \quad \Omega_1 \quad \mathcal{L}u^{n+1}_2 = f \quad \text{in} \quad \Omega_2
$$

$$
u^{n+1}_1 = u^n_2 \quad \text{on} \quad \partial\Omega_1 \cap \Omega_2 \quad u^{n+1}_2 = u^n_1 \quad \text{on} \quad \partial\Omega_2 \cap \Omega_1
$$

Algebraically, the global stationary (RAS) iteration becomes

$$
u^{n+1} = u^n + \left( \sum_{j=1}^{2} R_j^T D_j (R_j A R_j^T)^{-1} R_j \right) (f - A u^n)
$$

and the preconditioned system is given by

$$
\left( \sum_{j=1}^{2} R_j^T D_j (R_j A R_j^T)^{-1} R_j \right) A u = \left( \sum_{j=1}^{2} R_j^T D_j (R_j A R_j^T)^{-1} R_j \right) f
$$

$\mathbf{R}_j$ notation allows for global iteration, algebraic definition, overlapping subdomains.
1D example- Restriction, POU matrices

Given set of indices $\mathcal{N} = \{0, 1, 2, 3, 4\}$: partitioned into $\mathcal{N}_1 = \{0, 1, 2, 3\}$ and $\mathcal{N}_2 = \{2, 3, 4\}$, restriction and partition of unity matrices are given as

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

▶ Satisfies $I = \sum_{j=1}^{2} R_j^T D_j R_j$. 
Need for coarse correction

- Coarse corrections allows for global communication between all subdomains.
- Coarse correction (two-level methods) necessary for scalability for large number of subdomains.
- Generally, without coarse correction: Iterations scale with $N$. 
2-level RAS iteration: \( N \) Subdomains

Combine (multiplicatively) the 1-level RAS iteration

\[
M_{RAS,1}^{-1} = \sum_{j=1}^{N} R_j^T D_j (R_j A R_j^T)^{-1} R_j
\]

with the coarse approximation

\[
M_0^{-1} = R_0^T (R_0 A R_0^T)^{-1} R_0.
\]

and solve

\[
u^{n+\frac{1}{2}} = u^n + M_{RAS,1}^{-1} (f - A u^n),
\]

\[
u^{n+1} = u^{n+\frac{1}{2}} + M_0^{-1} (f - A u^{n+\frac{1}{2}}),
\]

\[\blacktriangleright\] \( R_j \): Correspond to overlapping subdomains.
The 2-level preconditioner for Krylov

Combine (additively) the 1-level RAS iteration

\[ M_{RAS,1}^{-1} = \sum_{j=1}^{N} R_j^T D_j (R_j A R_j^T)^{-1} R_j \]

with the coarse approximation

\[ M_0^{-1} = R_0^T (R_0 A R_0^T)^{-1} R_0. \]

to give

\[ M_{RAS,2}^{-1} = M_0^{-1} + M_{RAS,1}^{-1}. \]

and solve

\[ M_{RAS,2}^{-1} A u = M_{RAS,2}^{-1} f. \]
(Some) existing overlapping Schwarz coarse spaces

- Nicolaides: Piecewise constant by subdomain;
- Spectral spaces (eigenvalue problems): DtN, GenEO, SHEM (spectrally enriched MSFEM);
- Energy-minimizing spaces: GDSW, AGDSW, RGDSW;
- Multi-scale FEM: MsFEM
  - Numerically compute harmonic basis functions.
  - Used to approximate solution on coarse grid, but can use as DD coarse space!
Choice of coarse space

- Idea: want to take advantage of a-priori location of perforations (buildings/walls);
- Want robustness with respect to perforation size/location (even along subdomain interfaces);
- Want to choose a coarse space with approximation properties to improve convergence;
- Choose: Local harmonic basis functions occurring at intersection of a perforation with the coarse skeleton.
  - Think of as ’enriching’ MsFEM coarse space.
  - Works on nonoverlapping subdomains $\Omega_j$. 
Coarse-cell conforming triangulation

Mesh generation process:

- Larger $N \rightarrow$ more basis functions, larger coarse matrix;
- Triangulate after nonoverlapping coarse cell partitioning $\Omega'_j$;
- Overlap subdomains by layers of triangles for RAS.

2×2 subdomains  8×8 subdomains
Coarse grid nodes for coarse space basis functions

- Nonoverlapping skeleton: $\Gamma = \bigcup_{j \in \{1, \ldots, N\}} \partial \Omega_j'$
- $(e_k)_{k=1, \ldots, N_e}$: Partitioning of $\Gamma$; each “coarse edge” $e_k$ is an open planar segment;
- Set of coarse grid nodes: $\bigcup_{k=1, \ldots, N_e} \partial e_k$
- $(\phi_s)_{s \in \{1, \ldots, N_x\}}$: Locally harmonic basis functions for each coarse grid node.
- # of coarse grid nodes is automatically generated.
Basis functions: boundary conditions

For each coarse grid node \( x_s \), define \( g_s : \Gamma \rightarrow [0, 1] \) as: for \( i = 1, \ldots, N_x \),

\[
g_s(x_i) = \begin{cases} 
1, & s = i, \\ 
0, & s \neq i,
\end{cases}
\]

- \( g_s \) is linearly extended on the remainder of \( \Gamma \).
- Can also include higher-order polynomials on coarse edges.
Basis functions: Harmonic local solutions

For all nonoverlapping \( (\Omega_j')_{j \in \{1, \ldots, N\}} \) and \( s = 1, \ldots, N_x \), to obtain \( \phi_{s,j} = \phi_s|_{\Omega_j} \), solve

\[
\begin{align*}
-\Delta \phi_{s,j} &= 0 \quad \text{in } \Omega_j', \\
-\frac{\partial \phi_{s,j}}{\partial n} &= 0 \quad \text{on } \partial \Omega_j' \cap \partial \Omega_S, \\
\phi_{s,j} &= g_s \quad \text{on } \partial \Omega_j' \setminus \partial \Omega_S.
\end{align*}
\]

\( \triangleright \) supp(\( \phi_s \)) = \{ \bigcup_j \Omega_j' \mid x_s \text{ is a coarse grid node belonging to } \partial \Omega_j' \} \).

\( \triangleright \) Continuously, the coarse space is given by

\[ V_H = \text{span}\{\phi_s\}. \]
Approximation properties: Coarse approximation

Discretely, given

\[ M_0^{-1} = R_0^T (R_0 A R_0^T)^{-1} R_0. \]

the coarse approximation is the solution of

\[ u_H = M_0^{-1} f. \]

- Can use \( u_H \) as initial iterate for iteration, Krylov methods.
Experiment 1: Iterative RAS, L-shaped domain

- Provide iterative RAS results for preliminary L-shaped domain;
  - L-shaped domain: Square domain with one perforation;
  - Allows us to compare to analytical solution.
- Perform additional refinement at the singularity to improve convergence and FE error;
- Keep $N$ constant, vary $h$ and improve FE error;
- In spirit of iterative methods.
Numerical Results: Iterative RAS (L-shaped domain)

- SD error: Error from algebraic single domain FE solution;
- True error: Error from analytical true solution.
Edge refinement

> Orig. coarse grid nodes
> Additional edge refinement

- Improves coarse approximation;
- No changes to coarse skeleton $\Gamma$.
- Idea from MHM literature.
Numerical Results: Iterative RAS (L-shaped domain) Edge refinement

- Vary $H = \max_{k=1,\ldots,N_e} |e_k|$, keep $h$ constant;
- Edge refinement provides additional acceleration (better coarse approx., steeper slope).
Experiment 2: Iterative+Krylov, real data set

- Provide same iterative convergence curves as L-shaped domain;
- Also provide convergence curves for preconditioned GMRES;
- Multiple singularities and no analytical solution available.
Numerical Results: Iterative RAS (Real data)

SD error: Error from algebraic single domain FE solution;
True error: Error from fine FE solution.
Numerical Results: Krylov

SD error

True error
Experiment 3: Krylov Scalability, large real data set

- Want to show scalability:
- “Strong” scalability tests: Keep model domain and $h$ constant, vary $N$.

$\approx 300K$ DOFS in FE triangulation.
Numerical Results: Krylov (table)

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\text{min}$</th>
<th>$\frac{H}{20}$</th>
<th>$\text{dim. (rel)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>56</td>
<td>22</td>
<td>400 (16.0)</td>
</tr>
<tr>
<td>64</td>
<td>56</td>
<td>26</td>
<td>880 (10.9)</td>
</tr>
<tr>
<td>256</td>
<td>59</td>
<td>30</td>
<td>1912 (6.6)</td>
</tr>
<tr>
<td>1024</td>
<td>61</td>
<td>28</td>
<td>4253 (3.9)</td>
</tr>
</tbody>
</table>

- Relative dimension (rel): Compared to would-be homogeneous domain, $\frac{\text{dim}(R_0)}{(\sqrt{N}+1)^2}$.
- Relative dimension reduces as $N$ increases;
- Trefftz-like space produces scalable, accelerated iterations.
Closing Remarks

- We have presented a novel Trefftz-like coarse space that can be used to approximate the fine-scale solution;
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- We have presented a novel Trefftz-like coarse space that can be used to approximate the fine-scale solution;
- The space can also be used in combination with Schwarz methods to achieve fine-scale accuracy.
- Achieve fine-scale error in a small number of iterations, limited by finite element error;
- Krylov: Trefftz is Robust with respect to number of subdomains on a fixed total domain size, and provides an additional acceleration in terms of Krylov iteration count.
Closing Remarks

- We have presented a novel Trefftz-like coarse space that can be used to approximate the fine-scale solution;
- The space can also be used in combination with Schwarz methods to achieve fine-scale accuracy.
- Achieve fine-scale error in a small number of iterations, limited by finite element error;
- Krylov: Trefftz is Robust with respect to number of subdomains on a fixed total domain size, and provides an additional acceleration in terms of Krylov iteration count.
- However, the dimension of the Trefftz-like coarse space is large and controlled by the model geometry.
Funding Acknowledgement

This work has been supported by ANR Project Top-up (ANR-20-CE46-0005).

We also thank Métropole Nice Côte d’Azur for the given data.

Thank you for your time!