An introduction to domain decomposition methods

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IMPDE2023: research school on iterative methods for partial differential equations

Table of contents

First Part: Overview of classical DD methods.

- An overlapping method: the Schwarz method.
- Nonoverlapping/substructuring methods: Dirichlet-Neumann, Neumann-Neumann, FETI methods.
- Dichotomy between overlapping and substructuring DD methods.

Second Part: Additional topics.

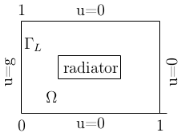
- Scalability and coarse spaces.
- The optimized Schwarz method and applications to multiphysics problems.
- Nonlinear preconditioning using DD methods.

Exercise session related to this mini-course

Goal of the exercise session: In this practical session, you will implement some classical domain decomposition methods to solve a model problem. The goal is to deepen the understanding of how these methods work, analyze their dependence on some parameters (e.g. overlap or Robin/relaxation parameters), verify the convergence results seen in the first lecture, and see how a Krylov method can accelerate convergence.

Model problem: Consider a heat diffusion problem posed on a square domain Ω. The adimensional temperature is equal to a step function g on the left edge, which assumes values $g = 0.3$ if $0 \leq y \leq 0.5 \wedge 0.9 \leq y \leq 1$ and $g = 1$ if $0.5 < y < 0.9$. On the rest of the boundary the temperature is fixed to zero. Inside the room there is a radiator whose temperature is equal to 50. The radiator is modeled by a source term $f(x, y) = 50$ if $(x, y) \in [0.4, 0.6] \times [0.4, 0.6]$ and zero otherwise. We want to find the temperature distribution inside the room described by the equation

$$\begin{aligned} -\Delta u &= f, && \text{in } \Omega, \\ u &= g, && \text{on } \Gamma_L, \\ u &= 0, && \text{on } \partial\Omega \setminus \Gamma_L. \end{aligned}$$



You are provided with Matlab/Octave scripts which you are partially required to complete.

Codes and exercise sheet available at https://github.com/vanzantom/

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Schwarz methods

Origins of DD methods: the alternating Schwarz method

Introduced by Schwarz in 1870 to improve Riemann's proof that $\int_\Omega |\nabla u|^2$ admits a minimizer on arbitrary domains of the form $\Omega = \Omega_1 \cup \Omega_2$.

$$-\Delta u = f \text{ in } \Omega,$$
$$u = g \text{ on } \partial\Omega.$$

Initial guesses u_1^0 and u_2^0.

$$\begin{aligned} -\Delta u_1^n &= f && \text{in } \Omega_1, & -\Delta u_2^n &= f && \text{in } \Omega_2,\\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1, & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2,\\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1, & u_2^n &= u_1^n && \text{on } \Gamma_2. \end{aligned}$$

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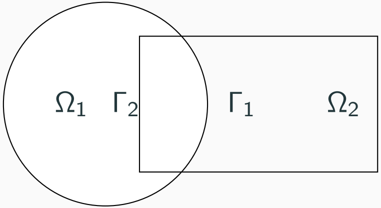

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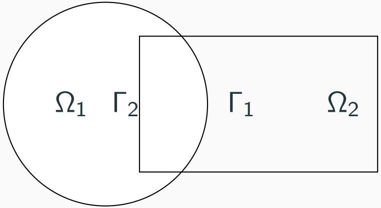

$$
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u &= g \text{ on } \partial\Omega.
\end{aligned}
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Initial guesses u_1^0 and u_2^0.

$$
\begin{aligned}
-\Delta u_1^n &= f && \text{in } \Omega_1, & -\Delta u_2^n &= f && \text{in } \Omega_2, \\
u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1, & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2, \\
u_1^n &= \color{blue}{u_2^{n-1}} && \text{on } \Gamma_1, & u_2^n &= \color{blue}{u_1^n} && \text{on } \Gamma_2.
\end{aligned}
$$

Error equation

Let $\{u_1^n\}_{n\geq 1}$, $\{u_2^n\}_{n\geq 1}$ be sequences of approximations generated by the Schwarz (or any stationary) method.

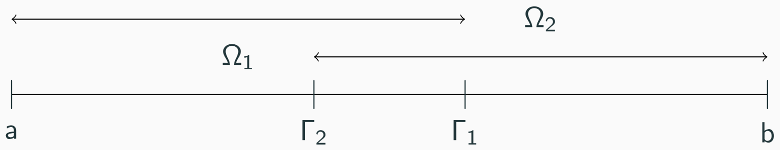
To study the convergence of $u_1^n \to u_{|\Omega_1}$ and $u_2^n \to u_{|\Omega_2}$ is sufficient to analyze how the quantities

$$e_1^n := u_{|\Omega_1} - u_1^n, \quad e_2^n := u_{|\Omega_2} - u_2^n,$$

converge to zero $e_j^n \to 0$, $j = 1, 2$.
In our setting, due to linearity, e_j^n satisfy

$$\begin{aligned} -\Delta e_1^n &= 0 && \text{in } \Omega_1, & -\Delta e_2^n &= 0 && \text{in } \Omega_2, \\ e_1^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_1, & e_2^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_2, \\ e_1^n &= e_2^{n-1} && \text{on } \Gamma_1, & e_2^n &= e_1^n && \text{on } \Gamma_2. \end{aligned}$$

The alternating Schwarz method: 1D example



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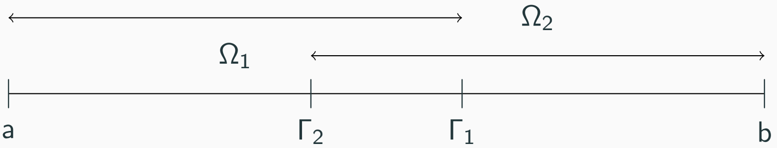
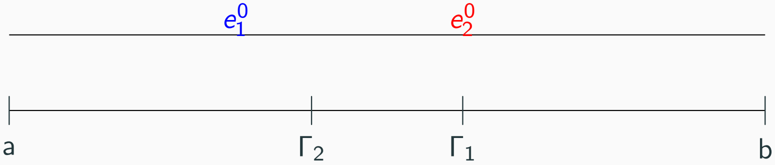


Consider $\Delta e = 0$, $e(a) = e(b) = 0$, and start with $e_1^0 = e_2^0 = 1$. Solution is $e = 0$.



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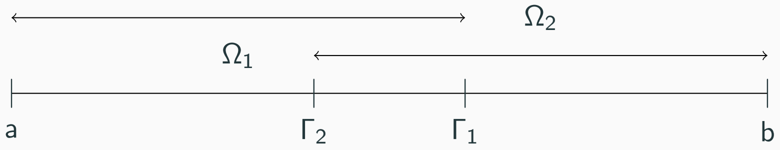

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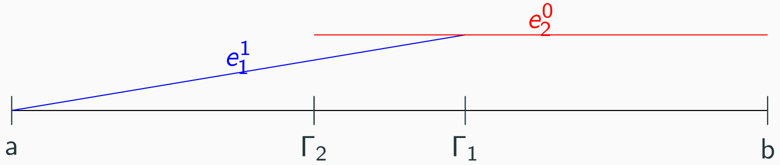
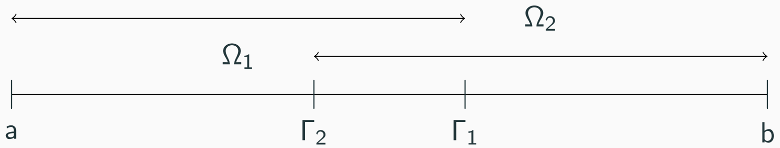


The alternating Schwarz method: 1D example



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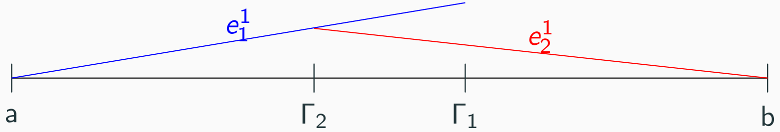

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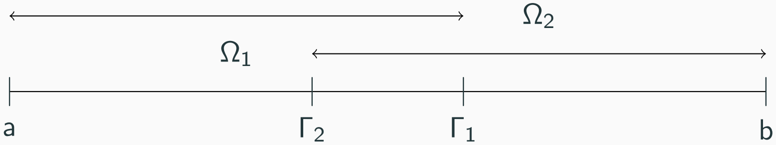

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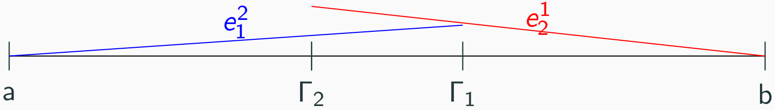

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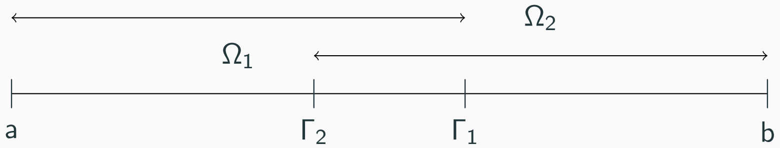

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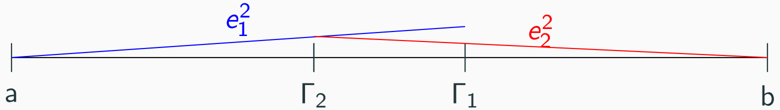

In the limit e_1^n and e_2^n tend to zero as $n \to \infty$.

The parallel Schwarz method: 1D example

A parallel version was introduced by P.L. Lions in 1989.

$$
\begin{aligned}
-\Delta u_1^n &= f && \text{in } \Omega_1, & -\Delta u_2^n &= f && \text{in } \Omega_2,\\
u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1, & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2,\\
u_1^n &= u_2^{n-1} && \text{on } \Gamma_1, & u_2^n &= u_1^{n-1} && \text{on } \Gamma_2.
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The parallel Schwarz method: 1D example

Error equation

$$
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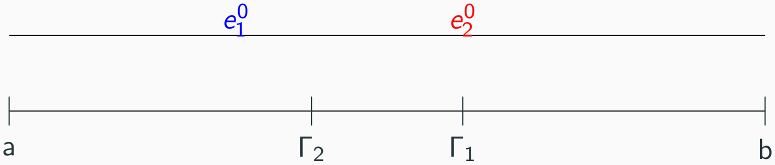
$$



The parallel Schwarz method: 1D example

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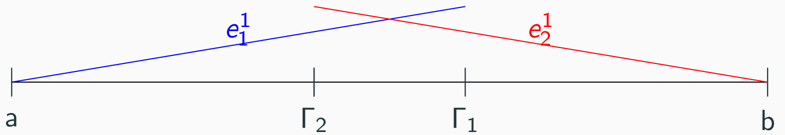

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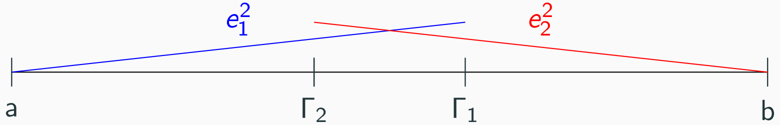

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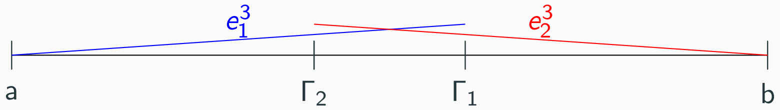


Convergence analysis in 1D

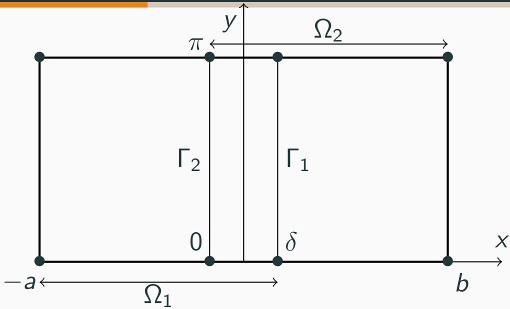
Define $v_{2,1}^n := e_2^n(\Gamma_1)$ and $v_{1,2}^n := e_1^n(\Gamma_2)$.
Then, $e_1^n(x) = v_{2,1}^{n-1}\frac{x-a}{\Gamma_1-a}$ and $e_2^n(x) = v_{1,2}^{n-1}\frac{b-x}{b-\Gamma_2}$.

$$\text{Further, } v_{1,2}^n := e_1^n(\Gamma_2) = v_{2,1}^{n-1}\frac{\Gamma_2-a}{\Gamma_1-a} = e_2^{n-1}(\Gamma_1)\frac{\Gamma_2-a}{\Gamma_1-a} = v_{1,2}^{n-2}\underbrace{\frac{b-\Gamma_1}{b-\Gamma_2}\frac{\Gamma_2-a}{\Gamma_1-a}}_{\rho}.$$

$$\implies v_{1,2}^n = \rho v_{1,2}^{n-2} \text{ with } \rho < 1 \implies \text{the Schwarz iterates converge to zero!}$$

- Conclusion: the larger the overlap $(\Gamma_1 - \Gamma_2)$ the fastest is the contraction!
- Remark: The analysis holds for any right hand side f and boundary condition g (sufficient to look at the error equation $e_j^n := u|_{\Omega_j} - u_j^n$.)

Convergence analysis in 2D in a simplified geometry



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e_1^n &= e_2^{n-1} && \text{on } \Gamma_1, & e_2^n &= e_1^{n-1} && \text{on } \Gamma_2.
\end{aligned}
$$

- Expand solutions in Fourier sine series: $e_j^n(x,y) = \sum_{j=1}^{\infty} \hat{e}_j^n(x,k)\sin(k\pi y)$.
- Insert expressions into $-\Delta e = 0$, we get $\sum_{j=1}^{\infty} \left(\partial_{xx} - k^2\right) \hat{e}_j^n(x,k)\sin(k\pi y) = 0$.
- Due to orthogonality, we can analyze the Schwarz algorithm frequency by frequency.

Remark: Same technique can be used to analyze the convergence for specific decompositions into many subdomains.

(Chaouqui et al, *On the scalability of classical one-level domain decomposition methods*, 2018).

Convergence analysis in 2D in a simplified geometry - II

$$\begin{aligned} \left(\partial_{xx} - k^2\right) \hat{e}_1^n(x,k) &= 0,\ x \in (-a,\delta), & \left(\partial_{xx} - k^2\right) \hat{e}_2^n(x,k) &= 0,\ x \in (0,b), && (1)\\ \hat{e}_1^n(-a,k) &= 0, & \hat{e}_2^n(b,k) &= 0, && (2)\\ \hat{e}_1^n(\delta,k) &= \hat{e}_2^{n-1}(\delta,k), & \hat{e}_2^n(0,k) &= \hat{e}_1^{n-1}(0,k). && (3) \end{aligned}$$

Using (1) and (2), subdomain solutions are

$$\hat{e}_1^n(x,k) = A^n(k)\sinh(\pi k(a+x)) \quad \text{and} \quad \hat{e}_2^n(x,k) = B^n(k)\sinh(\pi k(b-x)).$$

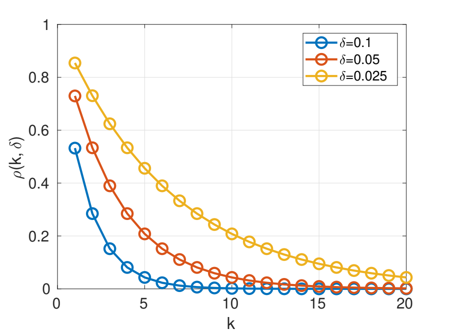
Using (3), we obtain

$$A^n(k) = \underbrace{\frac{\sin(k\pi a)}{\sinh(k\pi b)}\,\frac{\sinh(k\pi(b-\delta))}{\sinh(k\pi(a+\delta))}}_{\rho(k)} A^{n-2}(k).$$

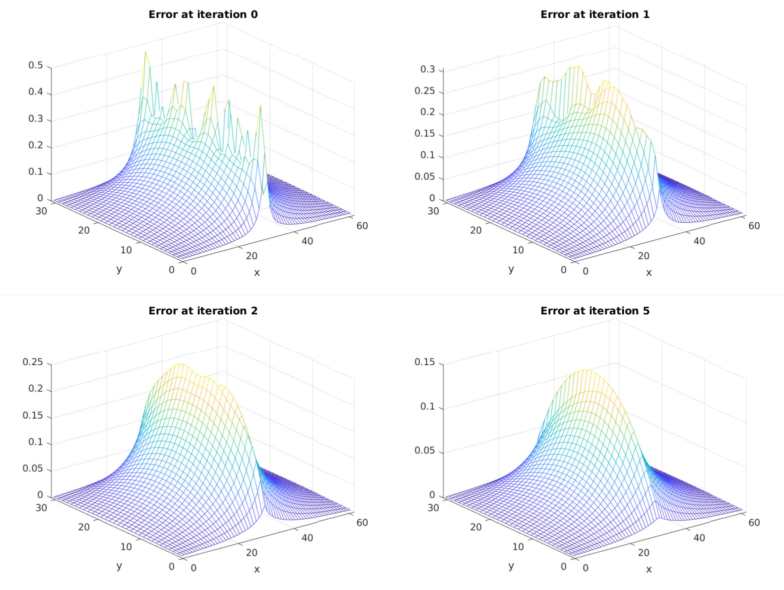
Remarks on the convergence of the Schwarz method

$$\rho(k,\delta) := \frac{\sin(k\pi a)}{\sinh(k\pi b)} \frac{\sinh(k\pi(b-\delta))}{\sinh(k\pi(a+\delta))}.$$

- The larger is δ the faster is the convergence.
- $\rho(k,0) = 1,\ \forall k \implies$ the Schwarz method does not convergence without overlap!
- the Schwarz method is an excellent smoother.



Visualization of the smoothing property



Generalization to many subdomains and algebraic formulation

Finite element mesh of size N_h.
$\Omega = \cup_{j=1}^{N} \Omega_j$, Ω_j are overlapping subdomains.
R_j are restriction operators to Ω_j (imagine boolean matrices in $\mathbb{R}^{N_{h,i} \times N_h}$. $R_j^\top$ are prolongation operators.
$\widetilde{R}_j$ are weighted restriction operators such that $\sum_{j=1}^{N} \widetilde{R}_j^\top R_j = I$.
$A_j = R_j A R_j^\top$ (local stiffness matrices)
• The discrete equivalent of the parallel Schwarz method is the Restricted Additive Schwarz method (Cai, Sarkis, 1999)

$$\boldsymbol{u}^n = \boldsymbol{u}^{n-1} + \sum_{j=1}^{N} \widetilde{R}_j^\top A_j^{-1} R_j(\boldsymbol{f} - A\boldsymbol{u}^{n-1}), \quad \text{Preconditioner is } M_{RAS}^{-1} := \sum_{j=1}^{N} \widetilde{R}_j^\top A_j^{-1} R_j$$

• Additive Schwarz preconditioner (Dryja, Widlund, 1987): $M_{AS}^{-1} := \sum_{j=1}^{N} R_j^\top A_j^{-1} R_j$.

Interpret RAS as the discretization of the parallel Schwarz method

The RAS method: a consistent discretization of the PSM to solve $A\boldsymbol{u} = \boldsymbol{f}$.

$$\begin{aligned}
\boldsymbol{u}^n &= \boldsymbol{u}^{n-1} + \textstyle\sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} R_j \left(\boldsymbol{f} - A\boldsymbol{u}^{n-1}\right) \\
&= \textstyle\sum_{j=1}^N \widetilde{R}_j^\top R_j \boldsymbol{u}^{n-1} + \sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} R_j \left(\boldsymbol{f} - A\boldsymbol{u}^{n-1}\right) && \left(\textstyle\sum_{j=1}^N \widetilde{R}_j^\top R_j = I\right) \\
&= \textstyle\sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} \left(A_j R_j \boldsymbol{u}^{n-1} + R_j \left(\boldsymbol{f} - A\boldsymbol{u}^{n-1}\right)\right) && \left(A_j^{-1} A_j = I\right) \\
&= \textstyle\sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} R_j \left(\boldsymbol{f} - A\left(I - R_j^\top R_j\right) \boldsymbol{u}^{n-1}\right) && \left(A_j = R_j A R_j^\top\right)
\end{aligned}$$

Assuming $A = (1/h^2)\text{diag}(-1, 2, -1)$

$$\begin{aligned}
R_1 \boldsymbol{u}^{n-1} &= (u_1^{n-1}, u_2^{n-1}, u_3^{n-1}, u_4^{n-1})^\top. \\
R_1^\top R_1 \boldsymbol{u}^{n-1} &= (u_1^{n-1}, u_2^{n-1}, u_3^{n-1}, u_4^{n-1}, 0, 0)^\top. \\
(I - R_1^\top R_1)\boldsymbol{u}^{n-1} &= (0, 0, 0, 0, u_5^{n-1}, u_6^{n-1})^\top. \\
A(I - R_1^\top R_1)\boldsymbol{u}^{n-1} &= (0, 0, 0, -u_5^{n-1}/h^2, x, x)^\top. \\
R_1(\boldsymbol{f} - A(I - R_1^\top R_1))\boldsymbol{u}^{n-1} &= (f_1, f_2, f_3, f_4 + u_5^{n-1}/h^2)^\top.
\end{aligned}$$

Interpret RAS as the discretization of the parallel Schwarz method

The RAS method: a consistent discretization of the PSM to solve $A\boldsymbol{u} = \boldsymbol{f}$.

$$
\begin{aligned}
\boldsymbol{u}^n &= \boldsymbol{u}^{n-1} + \textstyle\sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} R_j \left(\boldsymbol{f} - A\boldsymbol{u}^{n-1}\right) && \\
&= \textstyle\sum_{j=1}^N \widetilde{R}_j^\top R_j \boldsymbol{u}^{n-1} + \sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} R_j \left(\boldsymbol{f} - A\boldsymbol{u}^{n-1}\right) && (\textstyle\sum_{j=1}^N \widetilde{R}_j^\top R_j = I) \\
&= \textstyle\sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} \left(A_j R_j \boldsymbol{u}^{n-1} + R_j \left(\boldsymbol{f} - A\boldsymbol{u}^{n-1}\right)\right) && \left(A_j^{-1} A_j = I\right) \\
&= \textstyle\sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} R_j \left(\boldsymbol{f} - A\left(I - R_j^\top R_j\right)\boldsymbol{u}^{n-1}\right) && (A_j = R_j A R_j^\top)
\end{aligned}
$$

Assuming $A = (1/h^2)\mathrm{diag}(-1, 2, -1)$

$$
\begin{aligned}
R_1 \boldsymbol{u}^{n-1} &= (u_1^{n-1}, u_2^{n-1}, u_3^{n-1}, u_4^{n-1})^\top. \\
R_1^\top R_1 \boldsymbol{u}^{n-1} &= (u_1^{n-1}, u_2^{n-1}, u_3^{n-1}, u_4^{n-1}, 0, 0)^\top. \\
(I - R_1^\top R_1)\boldsymbol{u}^{n-1} &= (0, 0, 0, 0, u_5^{n-1}, u_6^{n-1})^\top. \\
A(I - R_1^\top R_1)\boldsymbol{u}^{n-1} &= (0, 0, 0, -u_5^{n-1}/h^2, x, x)^\top. \\
R_1(\boldsymbol{f} - A(I - R_1^\top R_1))\boldsymbol{u}^{n-1} &= (f_1, f_2, f_3, f_4 + u_5^{n-1}/h^2)^\top.
\end{aligned}
$$

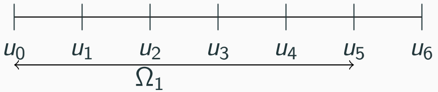
Interpret RAS as the discretization of the parallel Schwarz method

The RAS method: a consistent discretization of the PSM to solve $A\boldsymbol{u} = \boldsymbol{f}$.

$$
\begin{aligned}
\boldsymbol{u}^n &= \boldsymbol{u}^{n-1} + \textstyle\sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} R_j \left(\boldsymbol{f} - A\boldsymbol{u}^{n-1}\right) \\
&= \textstyle\sum_{j=1}^N \widetilde{R}_j^\top R_j \boldsymbol{u}^{n-1} + \sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} R_j \left(\boldsymbol{f} - A\boldsymbol{u}^{n-1}\right) && \left(\textstyle\sum_{j=1}^N \widetilde{R}_j^\top R_j = I\right) \\
&= \textstyle\sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} \left(A_j R_j \boldsymbol{u}^{n-1} + R_j \left(\boldsymbol{f} - A\boldsymbol{u}^{n-1}\right)\right) && \left(A_j^{-1} A_j = I\right) \\
&= \textstyle\sum_{j=1}^N \widetilde{R}_j^\top A_j^{-1} R_j \left(\boldsymbol{f} - A\left(I - R_j^\top R_j\right)\boldsymbol{u}^{n-1}\right) && \left(A_j = R_j A R_j^\top\right)
\end{aligned}
$$

Assuming $A = (1/h^2)\mathrm{diag}(-1, 2, -1)$

$$
\begin{aligned}
R_1 \boldsymbol{u}^{n-1} &= (u_1^{n-1}, u_2^{n-1}, u_3^{n-1}, u_4^{n-1})^\top. \\
R_1^\top R_1 \boldsymbol{u}^{n-1} &= (u_1^{n-1}, u_2^{n-1}, u_3^{n-1}, u_4^{n-1}, 0, 0)^\top. \\
(I - R_1^\top R_1)\boldsymbol{u}^{n-1} &= (0, 0, 0, 0, u_5^{n-1}, u_6^{n-1})^\top. \\
A(I - R_1^\top R_1)\boldsymbol{u}^{n-1} &= (0, 0, 0, -u_5^{n-1}/h^2, x, x)^\top. \\
R_1(\boldsymbol{f} - A(I - R_1^\top R_1))\boldsymbol{u}^{n-1} &= (f_1, f_2, f_3, f_4 + u_5^{n-1}/h^2)^\top.
\end{aligned}
$$



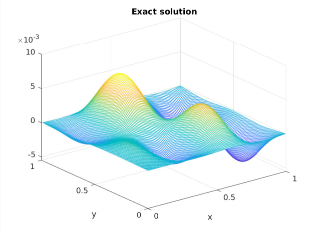
Comparison between RAS and AS

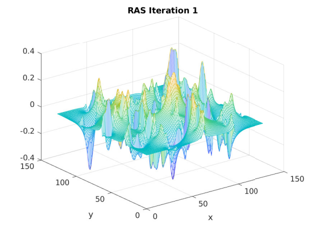
Property	AS	RAS
Works as stationary method	✗[1]	✓
Preserves symmetry	✓	✗
Condition number estimates	✓	✗
Convergence speed measured in # It.	✗	✓

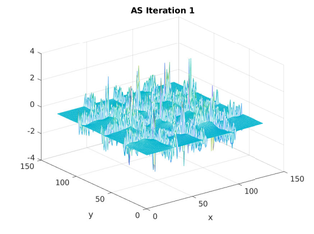
[1]Unless a damping parameter is suitable tuned

Comparison between RAS and AS

Property	AS	RAS
Works as stationary method	✗[1]	✓
Preserves symmetry	✓	✗
Condition number estimates	✓	✗
Convergence speed measured in # It.	✗	✓



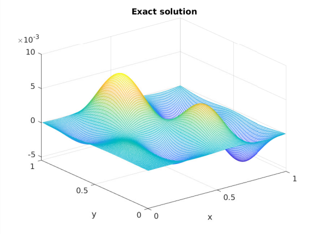


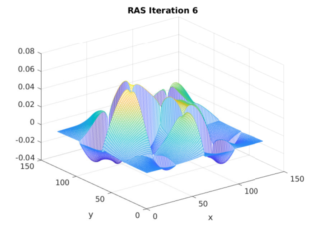


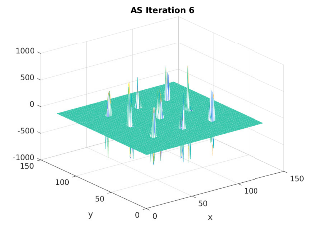
[1] Unless a damping parameter is suitable tuned

Comparison between RAS and AS

Property	AS	RAS
Works as stationary method	✗[1]	✓
Preserves symmetry	✓	✗
Condition number estimates	✓	✗
Convergence speed measured in # It.	✗	✓



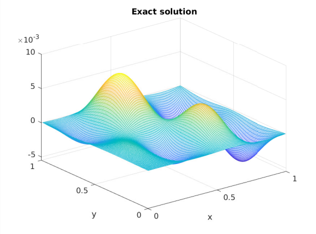


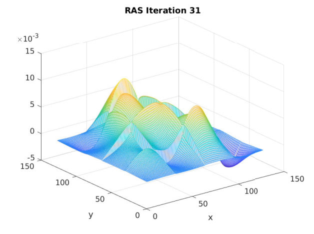


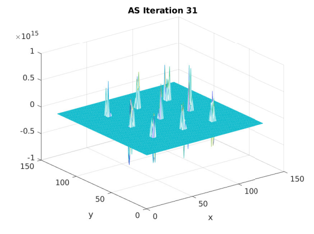
[1] Unless a damping parameter is suitable tuned

Comparison between RAS and AS

Property	AS	RAS
Works as stationary method	✗[1]	✓
Preserves symmetry	✓	✗
Condition number estimates	✓	✗
Convergence speed measured in # It.	✗	✓



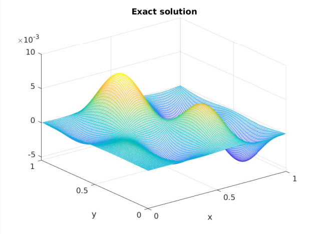


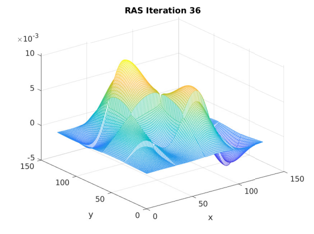


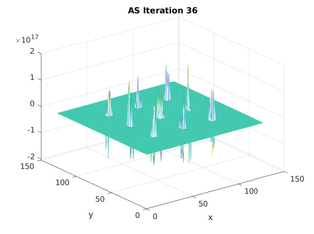
[1] Unless a damping parameter is suitable tuned

Comparison between RAS and AS

Property	AS	RAS
Works as stationary method	✗[1]	✓
Preserves symmetry	✓	✗
Condition number estimates	✓	✗
Convergence speed measured in # It.	✗	✓

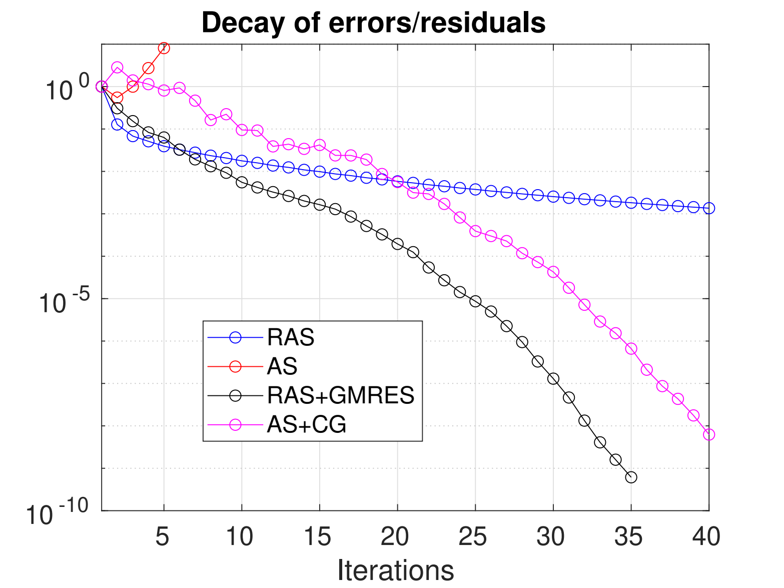






[1] Unless a damping parameter is suitable tuned

Convergence plots



Interlude on a condition number/eigenvalue distributions description of convergence

Contrary to common belief, neither a small condition number nor clustered eigenvalues guarantee fast convergence:

$$\text{CG: } \|\mathbf{u}^\star - \mathbf{u}_k\|_A \leq \min_{p\in\mathcal{P}_k : p(0)=1} \max_{\lambda_j \in \sigma(A)} |p(\lambda_j)| \|\mathbf{u}^\star - \mathbf{u}_0\|_A \leq 2\left(\frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1}\right)^k \|\mathbf{u}^\star - \mathbf{u}_0\|_A.$$

- First CG bound is sharp ($\exists$ a $\mathbf{u}_0$ such that bound is attained).
- Second bound is not sharp and sometimes even useless. (E.g. N small).
- Matrices with same condition number may exhibit very different convergence.
- Having several clustered eigenvalues is not equivalent to have a single eigenvalue.
- Convergence behaviour depends both on A and on $\mathbf{u}_0$.

Extensive discussion on the influence of eigenvalues distribution and condition number on CG/GMRES convergence in Section 5.6/5.7 in [Liesel, Strakos, 2012].

Link between stationary methods and preconditioners

Theorem (Stationary methods and preconditioned-Krylov methods)

Consider a splitting $A = M - N$ with M invertible, the corresponding stationary method, and a Krylov method minimizing the residual applied to the preconditioned system $M^{-1}A\mathbf{u} = M^{-1}\mathbf{f}$. Define the corresponding preconditioned residuals as

$$\mathbf{r}^n_{stat} := M^{-1}\mathbf{f} - M^{-1}A\mathbf{u}^n_{stat} \quad \text{and} \quad \mathbf{r}^n_{kry} := M^{-1}\mathbf{f} - M^{-1}A\mathbf{u}^n_{kry}.$$

Then we have that

$$\|\mathbf{r}^n_{kry}\|_2 \leq \|\mathbf{r}^n_{stat}\|_2 \ \text{ for any } n = 0, 1, 2, \ldots$$

A Krylov method minimizing the residual applied to $M^{-1}A\mathbf{u} = M^{-1}\mathbf{f}$ can never perform more iterations than a (convergent) stationary iterative method based on M.

Nonoverlapping/Substructuring methods

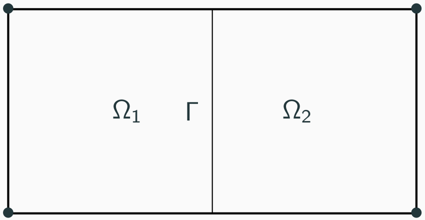
General idea behind nonoverlapping methods

Let $\Omega = \Omega_1 \cup \Omega_2$, with $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$.
The partial differential equation

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

can be equivalently be formulated as

$$\begin{aligned} -\Delta u_1 &= f && \text{in } \Omega_1, \\ -\Delta u_2 &= f && \text{in } \Omega_2, \\ u_1 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1, \\ u_2 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2, \\ u_1 &= u_2 && \text{on } \Gamma, \\ \frac{\partial u_1}{\partial n_1} &= \frac{\partial u_2}{\partial n_1} && \text{on } \Gamma, \end{aligned}$$



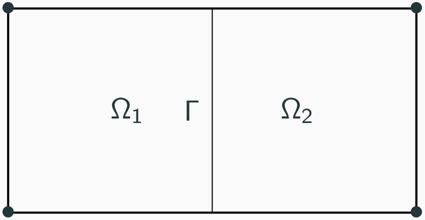
n_1 being the outward normal vector on Γ from Ω_1.

Dirichlet-Neumann method (Bjørstad & Widlund 1986)

Start from u_Γ^0.

$$
\begin{aligned}
-\Delta u_1^n &= f && \text{in } \Omega_1, & -\Delta u_2^n &= f && \text{in } \Omega_2,\\
u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1, & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2,\\
u_1^n &= u_\Gamma^{n-1} && \text{on } \Gamma, & \partial_x u_2^n &= \partial_x u_1^n && \text{on } \Gamma.
\end{aligned}
$$

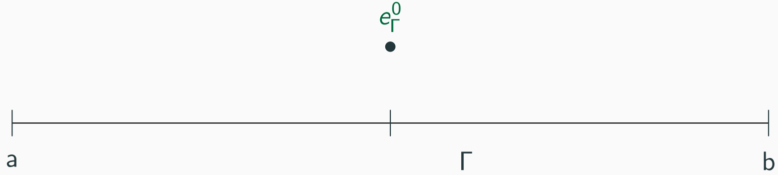
Update $u_\Gamma^n = \theta u_\Gamma^{n-1} + (1-\theta) u^n_{2,|\Gamma}$, $\theta \in [0,1)$.



Error equation

$$\begin{aligned} -\Delta e_1^n &= 0 && \text{in } \Omega_1, & -\Delta e_2^n &= 0 && \text{in } \Omega_2, \\ e_1^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_1, & e_2^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_2, \\ e_1^n &= e_\Gamma^{n-1} && \text{on } \Gamma, & \partial_x e_2^n &= \partial_x e_1^{n-1} && \text{on } \Gamma. \end{aligned}$$

Update $e_\Gamma^n = \frac{1}{2} e_\Gamma^{n-1} + \frac{1}{2} e_{2,|\Gamma}^n$.

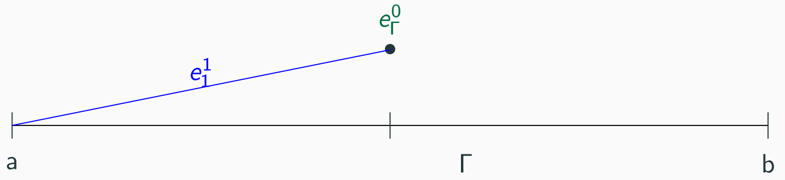


Dirichlet-Neumann for a symmetric 1D problem: $\theta = 1/2$.

Error equation

$$
\begin{aligned}
-\Delta e_1^n &= 0 && \text{in } \Omega_1, & -\Delta e_2^n &= 0 && \text{in } \Omega_2,\\
e_1^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_1, & e_2^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_2,\\
e_1^n &= e_\Gamma^{n-1} && \text{on } \Gamma, & \partial_x e_2^n &= \partial_x e_1^{n-1} && \text{on } \Gamma.
\end{aligned}
$$

Update $e_\Gamma^n = \frac{1}{2} e_\Gamma^{n-1} + \frac{1}{2} e_{2,|\Gamma}^n$.

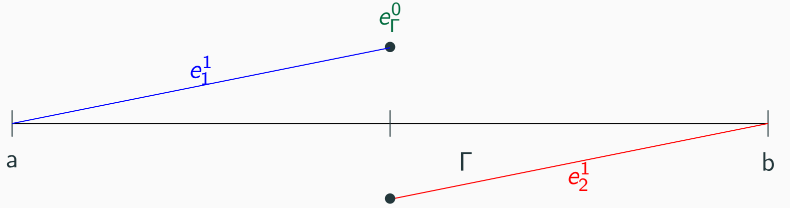


Dirichlet-Neumann for a symmetric 1D problem: $\theta = 1/2$.

Error equation

$$
\begin{aligned}
-\Delta e_1^n &= 0 && \text{in } \Omega_1, & -\Delta e_2^n &= 0 && \text{in } \Omega_2,\\
e_1^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_1, & e_2^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_2,\\
e_1^n &= e_\Gamma^{n-1} && \text{on } \Gamma, & \partial_x e_2^n &= \partial_x e_1^{n-1} && \text{on } \Gamma.
\end{aligned}
$$

Update $e_\Gamma^n = \frac{1}{2} e_\Gamma^{n-1} + \frac{1}{2} e^n_{2,|\Gamma}$.

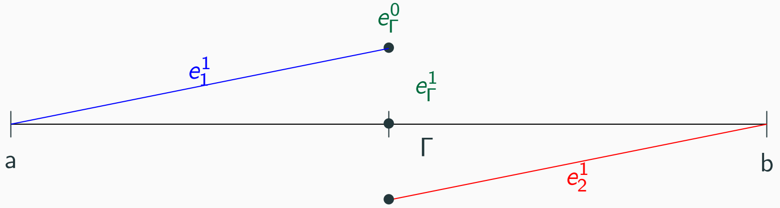


Dirichlet-Neumann for a symmetric 1D problem: $\theta = 1/2$.

Error equation

$$
\begin{aligned}
-\Delta e_1^n &= 0 && \text{in } \Omega_1, & -\Delta e_2^n &= 0 && \text{in } \Omega_2,\\
e_1^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_1, & e_2^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_2,\\
e_1^n &= e_\Gamma^{n-1} && \text{on } \Gamma, & \partial_x e_2^n &= \partial_x e_1^{n-1} && \text{on } \Gamma.
\end{aligned}
$$

Update $e_\Gamma^n = \frac{1}{2} e_\Gamma^{n-1} + \frac{1}{2} e_{2,|\Gamma}^n$.

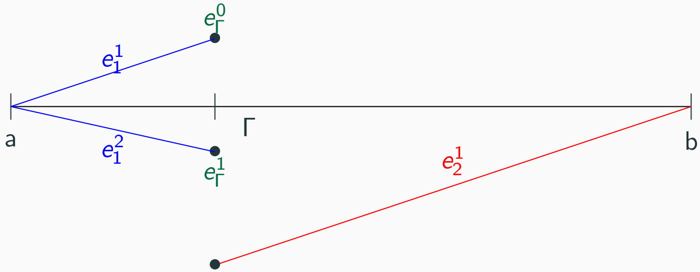


$e_\Gamma^1 = \frac{1}{2} e_\Gamma^0 + \frac{1}{2} e_{2,|\Gamma}^1 = 0 \implies$ the Dirichlet-Neumann method converges in two iterations!

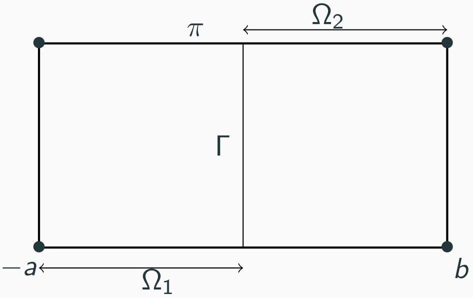
Dirichlet-Neumann for an unsymmetric 1D problem: $\theta = 1/2$.

$$
\begin{aligned}
-\Delta e_1^n &= 0 && \text{in } \Omega_1, & -\Delta e_2^n &= 0 && \text{in } \Omega_2,\\
e_1^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_1, & e_2^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_2,\\
e_1^n &= e_\Gamma^{n-1} && \text{on } \Gamma, & \partial_x e_2^n &= \partial_x e_1^{n-1} && \text{on } \Gamma.
\end{aligned}
$$

$$\text{Update } e_\Gamma^n = \frac{1}{2} e_\Gamma^{n-1} + \frac{1}{2} e^n_{2,|\Gamma}.$$

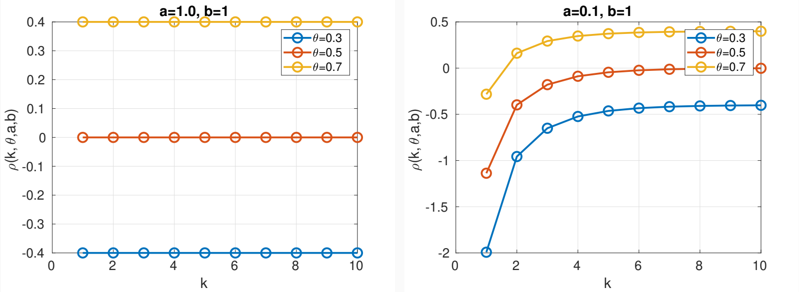


Convergence analysis in 2D in a simplified geometry

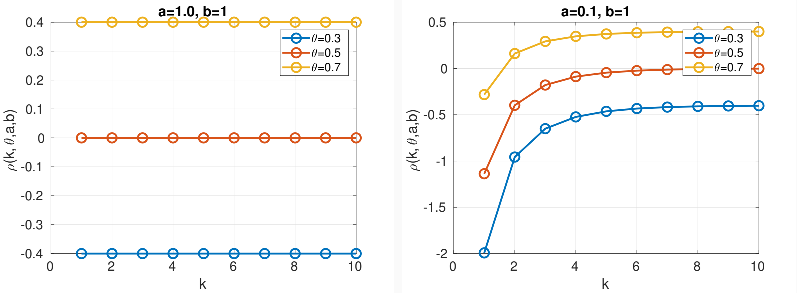


Using Fourier analysis we get

$$\rho(k, \theta, a, b) = \theta - (1 - \theta)\frac{\tanh(k\pi b)}{\tanh(k\pi a)}.$$



Remarks



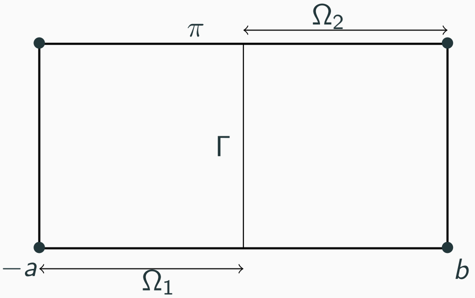
- If $a = b$, then $\rho = 2\theta - 1$. Hence, convergence in two iterations if $\theta = \frac{1}{2}$.
- It may diverge if θ is not chosen correctly.
- Convergence is sensible to asymmetry of the domain decomposition.
- If $\theta = \frac{1}{2}$, high frequencies convergence very fast $\implies$ good smoother!
- Not clear how to extend the method to many subdomains decompositions.

Neumann-Neumann (Bourgat, Glowinski, LeTallec, Vidrascu 1989)

Start from u_Γ^0.

$$
\begin{aligned}
-\Delta u_i^n &= f && \text{in } \Omega_i, & -\Delta \psi_i^n &= 0 && \text{in } \Omega_2,\\
u_i^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_i, & \psi_i^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_2,\\
u_i^n &= u_\Gamma^{n-1} && \text{on } \Gamma, & \partial_{n_i}\psi_i^n &= \partial_{n_1} u_1^n + \partial_{n_2} u_2^n && \text{on } \Gamma.
\end{aligned}
$$

Update $u_\Gamma^n = u_\Gamma^{n-1} \quad -\theta\left(\psi_{1,|\Gamma}^n + \psi_{2,|\Gamma}^n\right)$, $\theta \in [0,1)$.



Neumann-Neumann for an unsymmetric 1D problem: $\theta = 1/4$.

Error equation

$$
\begin{aligned}
\partial_{xx} e_i^n &= 0 && \text{in } \Omega_i, & \partial_{xx} \psi_i^n &= 0 && \text{in } \Omega_i,\\
e_i^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_i, & \psi_i^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_i,\\
e_i^n &= e_\Gamma^{n-1} && \text{on } \Gamma, & \partial_{n_i} \psi_i^n &= \partial_{n_1} e_1^{n-1} + \partial_{n_2} e_2^{n-1} && \text{on } \Gamma.
\end{aligned}
$$

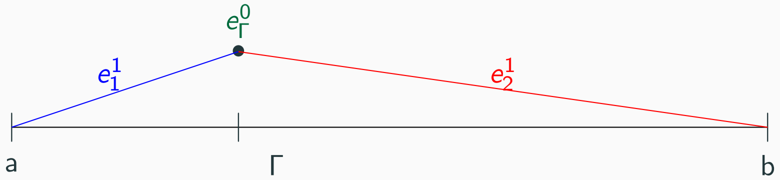
$$\text{Update } e_\Gamma^n = e_\Gamma^{n-1} - \theta \left(\psi_{1,|\Gamma}^n + \psi_{2,|\Gamma}^n \right).$$

Neumann-Neumann for an unsymmetric 1D problem: $\theta = 1/4$.

Error equation

$$\begin{aligned} \partial_{xx} e_i^n &= 0 && \text{in } \Omega_i, & \partial_{xx}\psi_i^n &= 0 && \text{in } \Omega_i,\\ e_i^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_i, & \psi_i^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_i,\\ e_i^n &= e_\Gamma^{n-1} && \text{on } \Gamma, & \partial_{n_i}\psi_i^n &= \partial_{n_1} e_1^{n-1} + \partial_{n_2} e_2^{n-1} && \text{on } \Gamma. \end{aligned}$$

$$\text{Update } e_\Gamma^n = e_\Gamma^{n-1} - \theta\left(\psi_{1,|\Gamma}^n + \psi_{2,|\Gamma}^n\right).$$

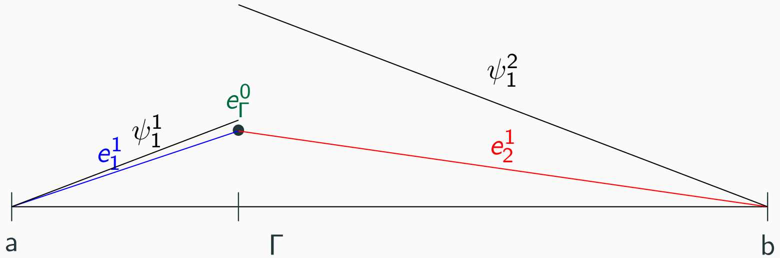


Neumann-Neumann for an unsymmetric 1D problem: $\theta = 1/4$.

Error equation

$$\begin{aligned}
\partial_{xx} e_i^n &= 0 && \text{in } \Omega_i, & \partial_{xx}\psi_i^n &= 0 && \text{in } \Omega_i,\\
e_i^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_i, & \psi_i^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_i,\\
e_i^n &= e_\Gamma^{n-1} && \text{on } \Gamma, & \partial_{n_i}\psi_i^n &= \partial_{n_1} e_1^{n-1} + \partial_{n_2} e_2^{n-1} && \text{on } \Gamma.
\end{aligned}$$

Update $e_\Gamma^n = e_\Gamma^{n-1} - \theta\left(\psi_{1,|\Gamma}^n + \psi_{2,|\Gamma}^n\right)$.



Neumann-Neumann for an unsymmetric 1D problem: $\theta = 1/4$.

Error equation

$$
\begin{aligned}
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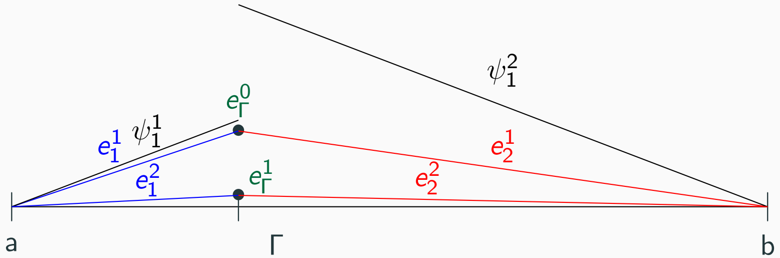
e_i^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_i, & \psi_i^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_i,\\
e_i^n &= e_\Gamma^{n-1} && \text{on } \Gamma, & \partial_{n_i}\psi_i^n &= \partial_{n_1} e_1^{n-1} + \partial_{n_2} e_2^{n-1} && \text{on } \Gamma.
\end{aligned}
$$

$$\text{Update } e_\Gamma^n = e_\Gamma^{n-1} - \theta \left(\psi_{1,|\Gamma}^n + \psi_{2,|\Gamma}^n \right).$$

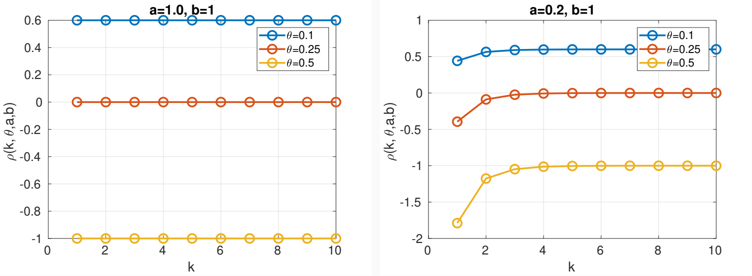


Convergence analysis in 2D in a simplified geometry

Using Fourier analysis,

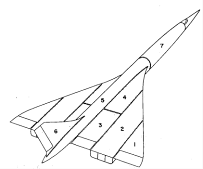
$$\rho(k, \theta, a, b) = 1 - \theta(\tanh(k\pi a) + \tanh(k\pi b))(\coth(k\pi a) + \coth(k\pi b)).$$

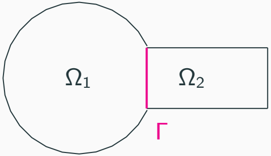
- If $a = b$, $\rho = 1 - 4\theta$, thus $\theta = \frac{1}{4}$ leads to convergence in two iterations!
- Choice of θ is very delicate if $a \gg b$ or $a \ll b$.
- If $\theta = \frac{1}{4}$, the Neumann-Neumann method is a good smoother.
- It may diverge for some values of θ!



Origins of nonoverlapping/substructuring methods and algebraic counterparts

Substructuring methods date back to the works of Cross (1930) and Przemieniecki (1963).





$$\begin{pmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_\Gamma \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_\Gamma \end{pmatrix}$$

Eliminating via Schur complement the interior degrees of freedom $\boldsymbol{u}_1$, $\boldsymbol{u}_2$,

$$S\boldsymbol{u}_\Gamma = \boldsymbol{\mu},$$

$S = S_1 + S_2$, with $S_j = A^j_{\Gamma\Gamma} - A_{\Gamma j}(A_{jj})^{-1} A_{j\Gamma}$,

$\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2$, $\boldsymbol{\mu}_j = \boldsymbol{f}^j_\Gamma - A^j_{\Gamma I}\left(A^j_{II}\right)^{-1} \boldsymbol{f}_j$.

Remark: A “substructured” system can be obtained at the continuous level using the Steklov-Poincaré operators.

Origins of nonoverlapping/substructuring methods and algebraic counterparts

Richardson iteration to solve $S\boldsymbol{u}_\Gamma = \boldsymbol{\mu}$,

$$\boldsymbol{u}_\Gamma^n = \boldsymbol{u}_\Gamma^{n-1} + P\left(\boldsymbol{\mu} - S\boldsymbol{u}_\Gamma^{n-1}\right).$$

- Dirichlet-Neumann is equivalent to choose $P = S_2^{-1}$
- Neumann-Neumann is equivalent to choose $P = S_1^{-1} + S_2^{-1}$.

Both preconditioners satisfy $\kappa(PS) \leq C$, with C independent on h (see Toselli-Widlund, Quarteroni-Valli).

Algebraic formulation of the Dirichlet-Neumann method

$$A_{II}^1 \mathbf{u}_1^{n+1} + A_{I\Gamma}^1 \mathbf{u}_\Gamma^n = \mathbf{f}_1, \qquad \text{Dirichlet problem in } \Omega_1,$$

$$\begin{pmatrix} A_{II}^2 & A_{I\Gamma}^2 \\ A_{\Gamma I}^2 & A_{\Gamma\Gamma}^2 \end{pmatrix} \begin{pmatrix} \mathbf{u}_2^{n+1} \\ \mathbf{u}_{2,\Gamma}^{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_2 \\ \mathbf{f}_\Gamma^2 + \mathbf{f}_\Gamma^1 - A_{I\Gamma}^1 \mathbf{u}_1^{n+1} - A_{\Gamma\Gamma}^1 \mathbf{u}_\Gamma^n \end{pmatrix}, \qquad \text{Neumann problem in } \Omega_2,$$

$$\mathbf{u}_\Gamma^{n+1} = \theta \mathbf{u}_\Gamma^n + (1-\theta)\mathbf{u}_{2,\Gamma}^{n+1}, \qquad \text{Update step.}$$

Eliminate $\boldsymbol{u}_1^{n+1}$ in the rhs of the Neumann problem using the Dirichlet problem,

$$\mathbf{f}_\Gamma^2 + \mathbf{f}_\Gamma^1 - A_{I\Gamma}^1 \mathbf{u}_1^{n+1} - A_{\Gamma\Gamma}^1 \mathbf{u}_\Gamma^n = \mathbf{f}_\Gamma^2 + \mathbf{f}_\Gamma^1 - A_{I\Gamma}^1 (A_{\Gamma\Gamma}^1)^{-1} \mathbf{f}^1 - (A_{\Gamma\Gamma}^1 \mathbf{u}_\Gamma^n - A_{I\Gamma}^1 (A_{\Gamma\Gamma}^1)^{-1} A_{\Gamma I}^1 \mathbf{u}_\Gamma^n) = \mathbf{f}_\Gamma^2 + \boldsymbol{\mu}_1 - S_1 \mathbf{u}_\Gamma^n.$$

Eliminate now $\boldsymbol{u}_2^{n+1}$ in the lhs of the Neumann problem via Schur complement,

$$A_{\Gamma I}^2 \mathbf{u}_2^{n+1} + A_{\Gamma\Gamma}^2 \mathbf{u}_{2,\Gamma}^{n+1} = (A_{\Gamma\Gamma}^2 - A_{\Gamma I}^2 (A_{II}^2)^{-1} A_{I\Gamma}^2) \mathbf{u}_{2,\Gamma}^{n+1} + A_{\Gamma I}^2 (A_{II}^2)^{-1} \mathbf{f}_2 = S_2 \mathbf{u}_{2,\Gamma}^{n+1} + A_{\Gamma I}^2 (A_{II}^2)^{-1} \mathbf{f}_2.$$

Thus,

$$S_2 \mathbf{u}_{2,\Gamma}^{n+1} = \boldsymbol{\mu} - S_1 \mathbf{u}_\Gamma^n,$$

which inserted into the update rule leads to

$$\boldsymbol{u}_\Gamma^n = \boldsymbol{u}_\Gamma^{n-1} + S_2^{-1} \left(\boldsymbol{\mu} - S \boldsymbol{u}_\Gamma^{n-1} \right).$$

Algebraic formulation of the Dirichlet-Neumann method

$$
\begin{aligned}
&A^1_{II}\mathbf{u}_1^{n+1} + A^1_{I\Gamma}\mathbf{u}_\Gamma^n = \mathbf{f}_1, && \text{Dirichlet problem in } \Omega_1,\\
&\begin{pmatrix} A^2_{II} & A^2_{I\Gamma} \\ A^2_{\Gamma I} & A^2_{\Gamma\Gamma} \end{pmatrix}\begin{pmatrix} \mathbf{u}_2^{n+1} \\ \mathbf{u}_{2,\Gamma}^{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_2 \\ \mathbf{f}_\Gamma^2 + \mathbf{f}_\Gamma^1 - A^1_{I\Gamma}\mathbf{u}_1^{n+1} - A^1_{\Gamma\Gamma}\mathbf{u}_\Gamma^n \end{pmatrix}, && \text{Neumann problem in } \Omega_2,\\
&\mathbf{u}_\Gamma^{n+1} = \theta\mathbf{u}_\Gamma^n + (1-\theta)\mathbf{u}_{2,\Gamma}^{n+1}, && \text{Update step.}
\end{aligned}
$$

Eliminate $\boldsymbol{u}_1^{n+1}$ in the rhs of the Neumann problem using the Dirichlet problem,

$$\mathbf{f}_\Gamma^2+\mathbf{f}_\Gamma^1-A^1_{I\Gamma}\mathbf{u}_1^{n+1}-A^1_{\Gamma\Gamma}\mathbf{u}_\Gamma^n = \mathbf{f}_\Gamma^2+\mathbf{f}_\Gamma^1-A^1_{I\Gamma}(A^1_{\Gamma\Gamma})^{-1}\mathbf{f}^1-(A^1_{\Gamma\Gamma}\mathbf{u}_\Gamma^n-A^1_{I\Gamma}(A^1_{\Gamma\Gamma})^{-1}A^1_{\Gamma I}\mathbf{u}_\Gamma^n) = \mathbf{f}_\Gamma^2+\boldsymbol{\mu}_1-S_1\mathbf{u}_\Gamma^n.$$

Eliminate now $\boldsymbol{u}_2^{n+1}$ in the lhs of the Neumann problem via Schur complement,

$$A^2_{\Gamma I}\mathbf{u}_2^{n+1} + A^2_{\Gamma\Gamma}\mathbf{u}_{2,\Gamma}^{n+1} = (A^2_{\Gamma\Gamma} - A^2_{\Gamma I}(A^2_{II})^{-1}A^2_{I\Gamma})\mathbf{u}_{2,\Gamma}^{n+1} + A^2_{\Gamma I}(A^2_{II})^{-1}\mathbf{f}_2 = S_2\mathbf{u}_{2,\Gamma}^{n+1} + A^2_{\Gamma I}(A^2_{II})^{-1}\mathbf{f}_2.$$

Thus,

$$S_2\mathbf{u}_{2,\Gamma}^{n+1} = \boldsymbol{\mu} - S_1\mathbf{u}_\Gamma^n,$$

which inserted into the update rule leads to

$$\boldsymbol{u}_\Gamma^n = \boldsymbol{u}_\Gamma^{n-1} + S_2^{-1}\left(\boldsymbol{\mu} - S\boldsymbol{u}_\Gamma^{n-1}\right).$$

FETI (Finite Element Tearing and Interconnecting) (Farhat, Roux 1991)

FETI is similar to a Neumann-Neumann method, but the Dirichlet and Neumann solves are inverted.

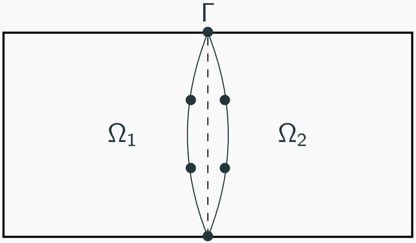
$$
\begin{aligned}
-\Delta u_i^n &= f && \text{in } \Omega_i, & -\Delta \psi_i^n &= 0 && \text{in } \Omega_2,\\
u_i^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_i, & \psi_i^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_2,\\
\frac{\partial u_i^n}{\partial n_i} &= (-1)^{1+i}\lambda_i^{n-1} && \text{on } \Gamma, & \psi_i^n &= u_1^n - u_2^n && \text{on } \Gamma.
\end{aligned}
$$

Update $\lambda^n = \lambda^{n-1} - \theta\left(\frac{\psi_1^n}{\partial n_1} + \frac{\psi_2^n}{\partial n_2}\right)$, $\theta \in [0,1)$.

$\Longrightarrow$ Exercise: study the convergence using Fourier analysis.

FETI - Algebraic formulation

$$\begin{pmatrix} A_{II}^1 & A_{I\Gamma}^1 & 0 & 0 & I \\ A_{\Gamma I}^1 & A_{\Gamma\Gamma}^1 & 0 & 0 & 0 \\ 0 & 0 & A_{II}^2 & A_{I\Gamma}^2 & 0 \\ 0 & 0 & A_{\Gamma I}^2 & A_{\Gamma\Gamma}^2 & 0 \\ 0 & I & 0 & -I & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I^1 \\ \boldsymbol{u}_\Gamma^1 \\ \boldsymbol{u}_I^2 \\ \boldsymbol{u}_\Gamma^2 \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I^1 \\ \boldsymbol{f}_\Gamma^1 \\ \boldsymbol{f}_I^2 \\ \boldsymbol{f}_\Gamma^2 \\ 0 \end{pmatrix}$$



Write system as $\begin{pmatrix} A & B^\top \\ B & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}$

FETI corresponds to solve $BA^{-1}B^\top \boldsymbol{\lambda} = \boldsymbol{\mu}$ with preconditioner $M = BSB^\top$, where $S = S_1 + S_2$.

At each Krylov iteration

- Multiplication by A^{-1} requires to solve two Neumann problems.
- Multiplication by S requires to solve two Dirichlet problems.
  More details in Klawoon, *FETI domain decomposition methods for second order partial differential equations*, 2006.

Dichotomy between overlapping and substructuring methods

!Nonoverlapping DD methods work on the substructured system $S\boldsymbol{u}_\Gamma = \boldsymbol{\mu}$.

!Overlapping DD methods act on the volume system $A\boldsymbol{u} = \boldsymbol{f}$.

Question: Can we formulate a substructured version of the parallel Schwarz method?

Remark: Only very few elements of u^{n-1} are needed to compute u^n!

Figure 1: Only the DOFs on the blue lines are needed to compute the next iterate!

Define $\overline{V}$ as the space of DOFs on the blue lines and introduce the restriction/prolongation operators $\overline{R}: V \to \overline{V}, \quad \overline{P}: \overline{V} \to V.$

We only suppose

$$\overline{R}\,\overline{P} = I_{\overline{V}} \quad \text{and} \quad \overline{R}M^{-1}A = \overline{R}M^{-1}A\overline{P}\,\overline{R}.$$

Dichotomy between overlapping and substructuring methods

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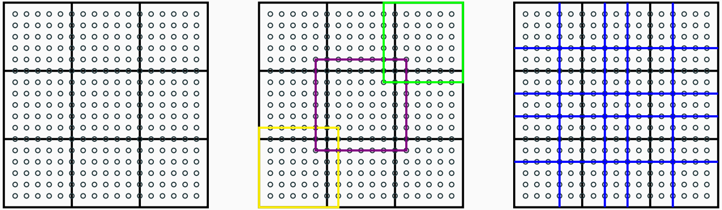


Figure 1: Only the DOFs on the blue lines are needed to compute the next iterate!

Define $\overline{V}$ as the space of DOFs on the blue lines and introduce the restriction/prolongation operators $\overline{R} : V \to \overline{V}, \quad \overline{P} : \overline{V} \to V.$

We only suppose

$$\overline{RP} = I_{\overline{V}} \quad \text{and} \quad \overline{R}M^{-1}A = \overline{R}M^{-1}A\overline{PR}.$$

Substructuring the RAS method I

Denote the linear operator of the RAS method with G^{RAS}, i.e.

$$u^n = u^{n-1} + M^{-1}(f - Au^{n-1}) =: G^{\text{RAS}}(u^{n-1}) \quad \forall n \in \mathbb{N}^*.$$

Then given a $v^0 \in \overline{V}$, we introduce the substructured iterative method

$$v^n = G^{\text{SRAS}}(v^{n-1}) \quad \text{where} \quad G^{\text{SRAS}}(v) := \overline{R} G^{\text{RAS}}(\overline{P} v).$$

Question: How are the convergence of the RAS and SRAS method linked?

Theorem (Equivalence between RAS and SRAS)

Assume that the operators $\overline{R}$ and $\overline{P}$ satisfy the assumptions. Given an initial guess $u^0 \in V$ and its substructured restriction $v^0 := \overline{R}u^0 \in \overline{V}$, define the sequences $\{u^n\}$ and $\{v^n\}$ such that

$$u^n = G^{RAS}(u^{n-1}), \quad v^n = G^{SRAS}(v^{n-1}).$$

Then, $\overline{R}u^n = v^n$ for every $n \geq 1$.

Substructing the RAS method II

Fixed point equation: $A_s \boldsymbol{v} = \boldsymbol{f}_s$, where $A_s = I - G$.

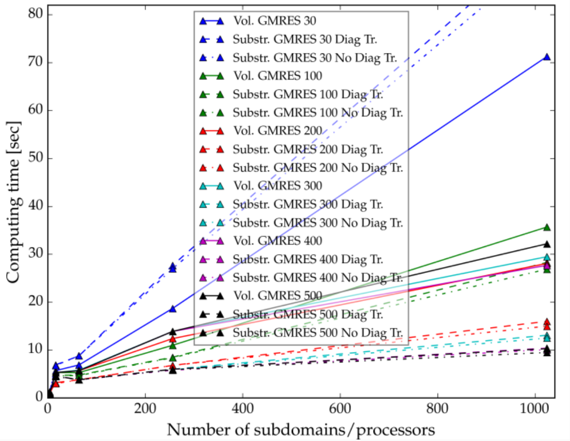
- ✓ Krylov acceleration is cheaper in a substructured form for Krylov methods that do not have short recurrences (e.g. GMRES).
- ✓ Less floating points operations.
- ✓ Less likely to run into memory issues (possibilities to use larger restarting parameters).
- ✓ New ideas and perspective to analyse two-level methods and derive coarse spaces[2][3].
- ✓ The substructured parallel Schwarz method can be defined at the continuous level.
- ✗ Equivalence requires exact local solves.

---

[2] Ciaramella, V.,*Substructured two-grid and multi-grid domain decomposition methods*, Num. Alg., 2022

[3] Ciaramella, V., *Spectral coarse spaces for the substructured parallel Schwarz method*, J. Sc. Comp., 2023

- $-\Delta u = f$, $f = 1$, $\Omega = (0,1)^2$.
- Weak scaling experiment: 256×256 nodes per subdomain. One subdomain per core.
- Experiments performed by Serge Van Criekingen, CNRS/IDRIS- Paris-Saclay.



Scalability and coarse spaces

Strong vs weak scalability[4]

Definition

An algorithm is said to be *strongly* scalable if, for a fixed <u>total</u> problem size, the elapsed time is inversely proportional to the number of cores.

Definition

An algorithm is said to be *weakly* scalable if, for a fixed problem size <u>per core</u>, the elapsed time is constant.

[4] *Introduction to domain decomposition methods*, Dolean et al. 2015

Strong vs weak scalability[4]

Definition

An algorithm is said to be *strongly* scalable if, for a fixed total problem size, the elapsed time is inversely proportional to the number of subdomains.

Definition

An algorithm is said to be *weakly* scalable if, for a fixed problem size per subdomain, the elapsed time is constant.

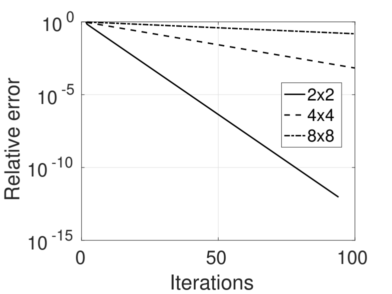
We may change *core* with *subdomain*.

- Strong scalability is not realistic to achieve.
- One-level domain decomposition methods are in general not weakly scalable.

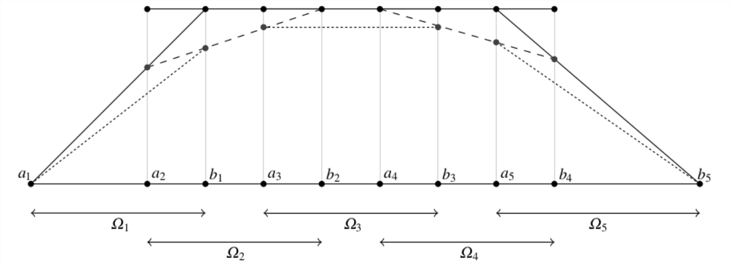
[4] *Introduction to domain decomposition methods*, Dolean et al. 2015

Domain decomposition methods are in general not weakly scalable

Convergence of the parallel Schwarz method applied to $-\Delta u = f$.



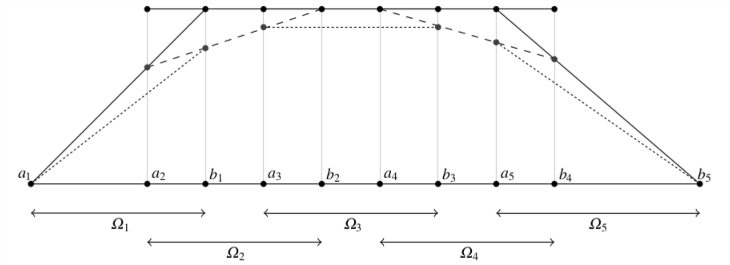
Intuition behind the lack of scalability



Generally, 1L-domain decomposition methods suffer the presence of "floating subdomain".

A subdomain Ω_j is said to be floating if $\partial\Omega_j \cap \partial\Omega = \emptyset$.

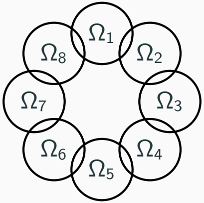
Intuition behind the lack of scalability

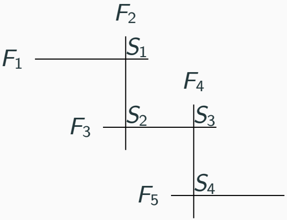


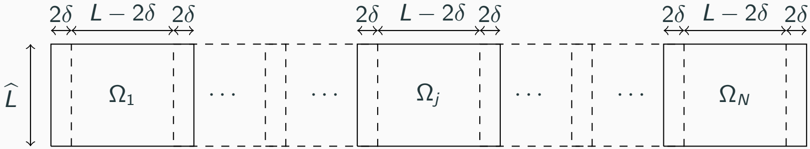
Generally, 1L-domain decomposition methods suffer the presence of "floating subdomain".

A subdomain Ω_j is said to be floating if $\partial\Omega_j \cap \partial\Omega = \emptyset$.

1L DD method may be weakly scalable if special geometries are involved







- Cancés, Maday, Stamm, *Domain decomposition for implicit solvation models*, 2013.
- Ciaramella, Gander, *Analysis of the parallel Schwarz method for growing chains of fixed-sixed subdomains: Part I-II-III*, 2018.
- Chaouqui, Ciaramella, Gander, V., *On the scalability of classical one-level domain decomposition methods*, 2018.
- Berrone, V., *Weak scalability of domain decomposition methods for discrete fracture networks*, 2023.

Coarse corrections

Idea: introduce a second level.

Given iteration $\boldsymbol{u}^n$:

$$\boldsymbol{r}_n = \boldsymbol{f} - A\boldsymbol{u}^n.$$
$$\boldsymbol{r}_c = R\boldsymbol{r}_n.$$
$$\boldsymbol{u}_c = A_c^{-1}\boldsymbol{r}_c.$$
$$\boldsymbol{u}_n = \boldsymbol{u}_n + P\boldsymbol{u}_c.$$

Components:

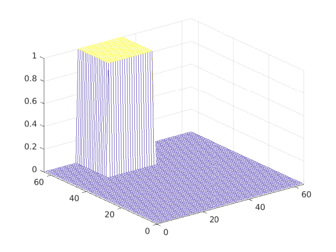
- Coarse space $V_c \subset V$.
- Restriction operator $R : V \to V_c$.
- Prolongator operator $P : V_c \to V$.
- Coarse matrix $A_c = RAP$.

Coarse spaces can be constructed *geometrically* (see Hardik's lecture tomorrow!) or *spectrally*.

Nicolaides coarse space 1987

Choose $P = [\psi_1, \psi_2, \ldots, \psi_N]$ where ψ_j is a constant function over subdomain Ω_j and zero everywhere else.

N. subdomains	4	16	64	128
RAS+Nicolaides	8	26	52	57

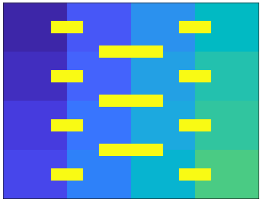


Coarse problem couples all subdomains $\implies$ inter-subdomains communication!

A coarse space can provide much more than just scalability

Remarks:

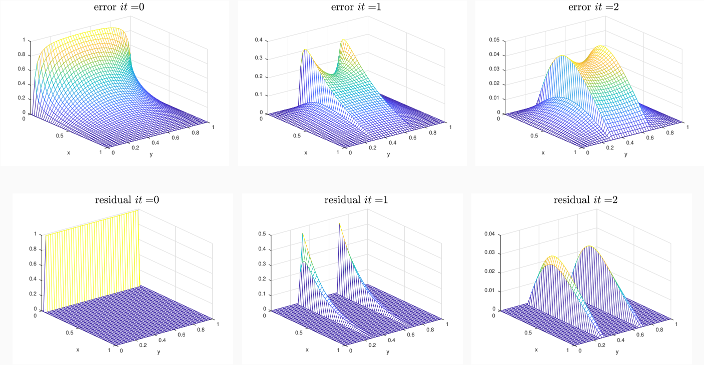
- A coarse space can make the method nilpotent. Such coarse spaces are called *complete*[5].
- A coarse space can make a DD method robust w.r.t. jumps in the diffusion coefficient. [6][7]



[5] Gander, Bo, *Complete, Optimal and Optimized Coarse Spaces for Additive Schwarz*, DDXXIV, 2018
[6] Gander, Loneland, *SHEM: an optimal coarse space for RAS and its multiscale approximation*, DDXXIII, 2016.
[7] Klawonn, Kuhn, Rheinbach, *Adaptive coarse spaces for FETI-DP in three dimensions with applications to heterogeneous diffusion problem*, DDXXIII, 2016

How to build efficient coarse spaces

General idea: V_c should contain the modes that the DD method does not handle efficiently.



The errors are mainly localized in the overlap and “harmonic” everywhere up to the interfaces. The residuals are zero everywhere up to the interfaces. These observations motivated different constructions of coarse space functions.

Some references

- SHEM: solves eigenvalue problems along the edges of the domain decompositions and extends harmonically.[8]
- GenEO: solves specific eigenvalues problems inside the subdomains. [9].
- GDSW: harmonic extension of the restriction of the null space of the Neumann matrix to the edges and vertices. [10]

---

[8] Gander, Loneland, *SHEM: an optimal coarse space for RAS and its multiscale approximation*, DDXXIII, 2016.

[9] Spillane, Dolean, Huaret, Nataf, Pechstein, Scheichl, *Abstract robust coarse spaces for systems of PDEs via generalized eigenvalues problems in the overlap*, Num. Mat., 2014

[10] Dorhmann, Klawonn, Widlund, *A Family of Energy Minimizing Coarse Spaces for Overlapping Schwarz Preconditioners*, DXVII, 2008.

A more recent approach

Theorem (Ciaramella, V., 2022)

Consider the substructured PSM, with $A = I - G$. Let $(\psi_j, \lambda_j)_{j=1}$ be eigenpairs of G. Then, if $V_c = span\{\psi_j\}_{j=1}^m$, then

- $\psi_j \in Kern(T)$, *where T is the two-level iteration matrix, $j = 1, \dots, m$.*
- $\rho(T) = |\lambda_{m+1}|$.

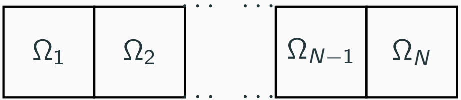
- Approximate numerically the slowest modes of G. [11].
- Compute them analytically (if you dare :)) [12]

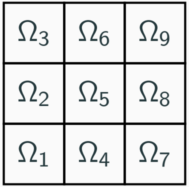
[11] Ciaramella, V.,*Spectral coarse spaces for the substructured parallel Schwarz method*, J. Sc. Comp., 2022

[12] Cuvelier, Gander, Halpern, *Fundamental coarse spaces components for Schwarz methods with crosspoints*, DDXXVI, 2022.

Extension to many subdomains case for nonoverlapping methods

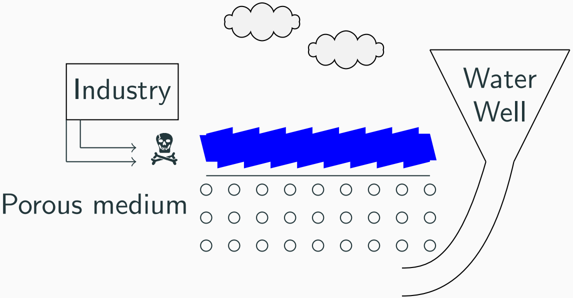
- Definition of the algorithm maybe not be unique (arbitrariness in the Dirichlet-Neumann).
- Neumann problems are not well-posed for floating subdomains (ad-hoc solutions).
- Both Dirichlet-Neumann and Neumann-Neumann are scalable for growing chains of fixed size subdomains (Chaouqui et al., 2018).
- Both Dirichlet-Neumann and Neumann-Neumann are not well-posed at the continuous level in presence of cross-points. (Chaudet-Dumas, Gander, 2022), (Chaudet-Dumas, Gander, 2023).





Multiphysics problems and the optimized Schwarz method

An instance of multiphysics problem



Mathematical model

Atmospheric Eq.

Navier-Stokes Eq.
Eq. for the contaminant

Darcy Eq.
Eq. for the contaminant

Monolitich approach

Let u_{At}, u_{Fl}, u_{Pm} be the unknowns of the Atmospheric, Fluid and Porous medium problem.

$$\text{Monolitich approach} \quad \begin{pmatrix} A_{\text{At}} & C_{\text{At,Fl}} & C_{\text{At,Pm}} \\ C_{\text{Fl,At}} & A_{\text{Fl}} & C_{\text{Fl,Pm}} \\ C_{\text{Pm,At}} & C_{\text{Pm,Fl}} & A_{\text{Pm}} \end{pmatrix} \begin{pmatrix} u_{\text{At}} \\ u_{\text{Fl}} \\ u_{\text{Pm}} \end{pmatrix} = \begin{pmatrix} f_{\text{At}} \\ f_{\text{Fl}} \\ f_{\text{Pm}} \end{pmatrix}$$

A monolithic approach is not always feasible:

- Physical phenomena can have very different time and space scales.
- Linear system has particular structure with blocks of different nature. Need for advanced problem-dependent solvers.

An instance of iterative decoupled approach

For $n = 1, 2, \dots$

$$\begin{pmatrix} A_{\text{At}} & 0 & 0 \\ 0 & A_{\text{Fl}} & 0 \\ 0 & 0 & A_{\text{Pm}} \end{pmatrix} \begin{pmatrix} u^n_{\text{At}} \\ u^n_{\text{Fl}} \\ u^n_{\text{Pm}} \end{pmatrix} = \begin{pmatrix} 0 & C_{\text{At,Fl}} & C_{\text{At,Pm}} \\ C_{\text{Fl,At}} & 0 & C_{\text{Fl,Pm}} \\ C_{\text{Pm,At}} & C_{\text{Pm,Fl}} & 0 \end{pmatrix} \begin{pmatrix} u^{n-1}_{\text{At}} \\ u^{n-1}_{\text{Fl}} \\ u^{n-1}_{\text{Pm}} \end{pmatrix} + \begin{pmatrix} f_{\text{At}} \\ f_{\text{Fl}} \\ f_{\text{Pm}} \end{pmatrix}$$

✓ Possibility to recycle ad-hoc solvers and codes for each subproblem.

✗ Convergence can depend badly on physical parameters.

Idea: Use the DD machinery to develop efficient and robust decoupled solvers.

An instance of iterative decoupled approach

For $n = 1, 2, \ldots$

$$\begin{pmatrix} A_{\mathrm{At}} & 0 & 0 \\ 0 & A_{\mathrm{Fl}} & 0 \\ 0 & 0 & A_{\mathrm{Pm}} \end{pmatrix} \begin{pmatrix} u^n_{\mathrm{At}} \\ u^n_{\mathrm{Fl}} \\ u^n_{\mathrm{Pm}} \end{pmatrix} = \begin{pmatrix} 0 & C_{\mathrm{At,Fl}} & C_{\mathrm{At,Pm}} \\ C_{\mathrm{Fl,At}} & 0 & C_{\mathrm{Fl,Pm}} \\ C_{\mathrm{Pm,At}} & C_{\mathrm{Pm,Fl}} & 0 \end{pmatrix} \begin{pmatrix} u^{n-1}_{\mathrm{At}} \\ u^{n-1}_{\mathrm{Fl}} \\ u^{n-1}_{\mathrm{Pm}} \end{pmatrix} + \begin{pmatrix} f_{\mathrm{At}} \\ f_{\mathrm{Fl}} \\ f_{\mathrm{Pm}} \end{pmatrix}$$

- ✓ Possibility to recycle ad-hoc solvers and codes for each subproblem.
- ✗ Convergence can depend badly on physical parameters.

Idea: Use the DD machinery to develop efficient and robust <u>decoupled</u> solvers.

Spatial decompoisition is provided by the physics itself and the number of subdomains is limited.

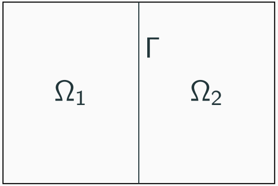
Natural DD framework involves nonoverlapping subdomain:[13]

Dirichlet-Neumann can be very efficient in a few regimes but it is not robust.
Optimized Schwarz methods are more efficient due to the possibility of optimizing the transmission conditions.

[13]About overlapping approaches see ICDD method of Discacciati, Gervasio, and Quarteroni, SIAM J. Control Optim. (2013).

A prototype example: diffusion equation with discontinuous coefficient.

$$
\begin{aligned}
-\nu_1 \Delta u_1 &= f \quad \text{in } \Omega_1,\\
-\nu_2 \Delta u_2 &= f \quad \text{in } \Omega_2,\\
u_1 &= u_2 \quad \text{on} \Gamma,\\
\nu_1 \partial_x u_1 &= \nu_2 \partial_x u_2 \quad \text{on} \Gamma.
\end{aligned}
$$



Equivalent but more effective transmission conditions:

$$
\begin{aligned}
(\nu_1 \partial_x + p_1) u_1 &= (\nu_2 \partial_x + p_1) u_2 \quad \text{on } \Gamma,\\
(-\nu_2 \partial_x + p_2) u_2 &= (-\nu_1 \partial_x + p_2) u_1 \quad \text{on } \Gamma.
\end{aligned}
$$

$$
\begin{aligned}
-\nu_1 \Delta u_1^n = f \quad \text{in } \Omega_1, \quad (\nu_1 \partial_x + p_1) u_1^n &= (\nu_2 \partial_x + p_1) u_2^{n-1} \quad \text{on } \Gamma\\
-\nu_2 \Delta u_2^n = f \quad \text{in } \Omega_2, \quad (-\nu_2 \partial_x + p_2) u_2^n &= (-\nu_1 \partial_x + p_2) u_1^{n-1} \quad \text{on } \Gamma.
\end{aligned}
$$

Fourier analysis: find the optimized transmission conditions

Remark: We could use even more general transmission conditions! Laplace-beltrami on the interface, nonlocal operators..

Using Fourier analysis, one obtains the convergence factor

$$\rho(k, p_1, p_2) = \left| \frac{\nu_2 |k| - p_1}{\nu_1 |k| + p_1} \frac{\nu_1 |k| - p_2}{\nu_2 |k| + p_2} \right|.$$

Rescaling $p_1 = \nu_2 p$ and $p_2 = \nu_1 q$ for a $p, q \in \mathbb{R}$, we solve for

$$(p^\star, q^\star) := \mathrm{argmin}_{p,q \in \mathbb{R}^+} \max_{k \in [k_{\min}, k_{\max}]} \rho(k, p, q).$$

Optimized Schwarz methods take advantage of heterogeneities

Define $\lambda := \frac{\nu_1}{\nu_2}$, it has been proven [Dubois, Gander, Num. Alg., 2015]

$$\text{If } \lambda \geq 1 \qquad \max_{k \in [k_{\min}, k_{\max}]} \rho(k, p^\star, q^\star) = \frac{1}{\lambda} - \frac{4(\lambda+1)}{\lambda(\lambda-1)} \sqrt{\frac{k_{\min}}{\pi}} h^{\frac{1}{2}} + O(h).$$

$$\text{If } \lambda < 1 \qquad \max_{k \in [k_{\min}, k_{\max}]} \rho(k, p^\star, q^\star) = \lambda - \frac{4(\lambda+1)\lambda}{1-\lambda} \sqrt{\frac{k_{\min}}{\pi}} h^{\frac{1}{2}} + O(h).$$

- Convergence is faster for $\lambda \lll 1$ and $\lambda \ggg 1$.
- Mesh independent convergence for $\lambda \neq 1$.
- Compare with classical Schwarz which needs a coarse space in order to be robust (see yesterday lectures).
- Extension to general second order elliptic PDEs [Gander, V., SISC, 2019].
- Extension to decompositions not aligned with the discontinuities [Gu, Kwok, J. Sci. comp., 2021]

Derivation of optimized transmission conditions

- Fluid-structure problems: [Badia, Nobile, Vergara, 2008], [Badia, Nobile, Vergara, 2009], [Gerardo-Giorda, Nobile, Vergara, 2010]
- Second order PDEs: [Gander-Dubois, 2015], [Gander, V., 2019].
- Wave-diffusion coupling: [Gander, V., 2018], [Chouly, Klein, 2021].
- Ocean-atmosphere coupling: [Lemaire, Blayo, Debreu, 2015], [Thery, Pelletier, Lemaire, Blayo, 2021].
- Electromagnetic problems: [Dolean, Gander, Veneros, Zhang, 2016].
- Stokes-Darcy coupling: [Discacciati, Quarteroni, Valli 2007],[Discacciati, Gerardo Giorda, 2018], [Cao, Gunzburger, He, Wang, 2011,2014], [Gander, V., 2019], [Phuong Hoang, Lee, 2021], [Discacciati, V., 2023]...

Nonlinear preconditioning

What does it mean to “precondition” a nonlinear system?

One possible definition[14]:
“The nonlinear system is transformed into a new nonlinear system, which has the same solution as the original system. For certain applications the non linearities of the new function are more balanced and, as a result, the (inexact) Newton method converges more rapidly.”

$$\begin{aligned} G(F(u)) &= 0, \\ F(H(y)), \quad u &= H^{-1}(y), \end{aligned}$$

$$\begin{aligned} M^{-1}Au = M^{-1}b, &\quad \text{(left prec.)}, \\ AP^{-1}y = b, \quad Pu = y &\quad \text{(right prec.)}. \end{aligned}$$

See also [15] for a review with an historical flavour.

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[14] X.-C. Cai and D. E. Keyes. *Nonlinearly preconditioned inexact Newton algorithms, 2002*
[15] M. J. Gander, *On the origins of linear and nonlinear preconditioning*, 2017

The RASPEN method[16]

Question: How to define the RAS method for nonlinear problems?

Answer: Introduce operators G_j such that $G_j(u)$ is the solution of

$$R_j F(P_j G_j(u) + (I - P_j R_j)u) = 0.$$

Sanity check: If $F(u) = Au - f$, then $G_j(u^{n-1}) = A_j^{-1} R_j (f - A(I - P_j R_j) u^{n-1})$, i.e. a subdomain solution with right hand side f and Dirichlet BC. given by u^{n-1}.

Definition: The nonlinear RAS method reads

$$u^n = \sum_{j=1}^{N} \widetilde{P}_j G_j(u^{n-1}), \quad \forall n \in \mathbb{N}^*.$$

. RASPEN= Restricted Additive Schwarz Preconditioning Exact Newton.

[16] Dolean et al, *Nonlinear Preconditioning: How to Use a Nonlinear Schwarz Method to Precondition Newton's Method*, 2016.

Stationary iterative methods

Linear case

$$u^n = u^{n-1} + M^{-1}(f - Au^{n-1})$$

Nonlinear case

$$u^n = \sum_{j=1}^{N} \widetilde{P}_j G_j(u^{n-1}).$$

If $\{u^n\}_{n\in\mathbb{N}^*}$ converges, i.e. $u^n \to u^*$, then u^* satisfies

$$u^* = u^* + M^{-1}(f - Au^*).$$

$$u^* = \sum_{j=1}^{N} \widetilde{P}_j G_j(u^*).$$

$$M^{-1}Au^* = M^{-1}f.$$

$$\mathcal{F}(u^*) := u^* - \sum_{j=1}^{N} \widetilde{P}_j G_j(u^*) = 0.$$

$\to$ apply a Krylov method!

$\to$ apply Newton's method!

Summary

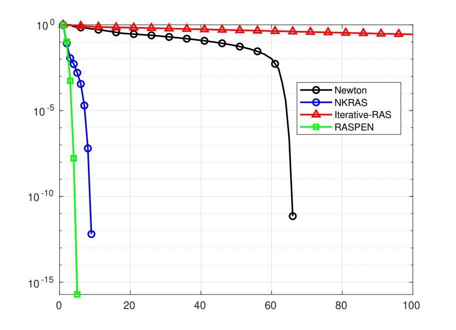
Suppose you aim to solve $F(u) = 0$.
Question: How can you use a DD method for a nonlinear problem?

- As a nonlinear iterative method.
- As a preconditioner for the Jacobian system inside a Newton's method (notation Newton-Krylov-DD). Several works from Klawonn et al. (2014,2016).
- As a right preconditioner for Newton's method, e.g. NKS-RAS by Cai et al., (2011,2018).
- As a left preconditioners for Newton's method, e.g. the RASPEN method (2016), the ORASPEN (2020).

Numerical examples: Forchheimer's equation

$$q(-\lambda(x)u(x)'))' = f(x) \quad \text{in } \Omega,$$
$$u(0) = 1 \quad \text{and} \quad u(1) = e,$$

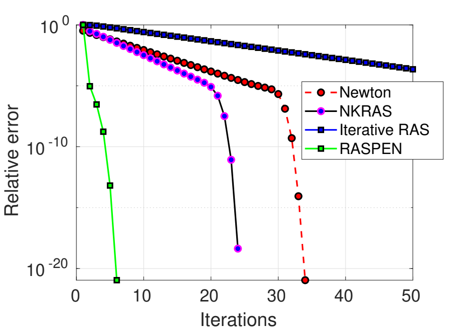
Parameters:
$q(y) := \text{sign}(y)\frac{-1+\sqrt{1+4|y|}}{2}$, $\lambda(x) = 2 + \cos(5\pi x)$, $f(x) = 50\sin(5\pi x)e^x$. $N_h = 10^3$, $N = 5$, $\delta = 4h$, $u^0 = 10^5$.



Numerical examples: Nonlinear diffusion equation

$$
\begin{aligned}
-\nabla \cdot \left((1+u^2)\nabla u\right) &= 1, \quad \text{in } (0,1)^2 =: \Omega,\\
u &= 0, \quad \text{on } \partial\Omega.
\end{aligned}
$$

Parameters: $N_h \approx 10^3$, $N = 4$, $\delta = 4h$, $u^0 = 10^5$.



Thank you!

Codes and exercise sheet available at https://github.com/vanzantom/

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