Preconditioning the stage equations of implicit Runge Kutta methods for parabolic PDEs

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Abstract

When using Solving parabolic PDEs, we need to discretize in both space and time and for the time domain and implicit Runge-Kutta methods are a common choice for the latter. There, solving the stage equations is often the computational bottleneck, as the dimension of the stage equations Mk = b for an s-stage Runge-Kutta method becomes sn where the spatial discretization dimension n can be very large. Hence the solution process often requires the use of iterative solvers, whose convergence can be less than satisfactory. Moreover, due to the structure of the stage equations, the matrix M does not necessarily inherit any of the preferable properties of the spatial operator, making GMRES the go-to solver and hence there is a need for a preconditioner. Recently in (1) and also (2,3) a new block preconditioner was proposed and numerically tested with promising results.

Using spectral analysis and the particular structure of M, we study the properties of this class of preconditioners, focusing on the eigen properties of the preconditioned system, and we obtain interesting results for the eigenvalues of the preconditioned system for a general Butcher matrix. In particular, for low number of stages, i.e., s = 2,3, we obtain explicit formulas for the eigen properties of the preconditioned system and for general s we can explain and predict the characteristic features of the spectrum of the preconditioned system observed in (2). As the eigenvalues alone are known to **not** be sufficient to predict the GMRES convergence behavior in general, we also focus on the eigenvectors, which altogether allows us to give descriptive bounds of the GMRES convergence behavior for the preconditioned system.

We then numerically optimize the Butcher tableau for the performance of the entire solution process, rather than only the order of convergence of the Runge-Kutta method. To do so requires careful balancing of the numerical stability of the Runge-Kutta method, its order of convergence and the convergence of the iterative solver for the stage equations. We illustrate our result on test problem and then outline possible generalizations.

References

(1) M. Neytcheva, O. Axelsson. Numerical solution methods for implicit Runge-Kutta methods of arbitrarily high order. Proceedings of ALGORITHMY 2020, ISBN : 978-80-227-5032-5, 2020.
(2) M. M. Rana, V. E. Howle, K. Long, A. Meek, W. Milestone. A New Block Preconditioner

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for Implicit Runge-Kutta Methods for Parabolic PDE Problems. SIAM Journal on Scientific Computing, vol (43): S475–S495, 2021.
(3) M. R. Clines, V. E. Howle, K. R. Long. Efficient order-optimal preconditioners for implicit Runge-Kutta and Runge-Kutta-Nystrom methods applicable to a large class of parabolic and hyperbolic PDEs. arXiv: https://arxiv.org/abs/2206.08991, 2022.

Randomization to speed-up Krylov subspace methods

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Abstract

The natural basis of the Krylov subspace is expected to be ill-conditioned, as the vectors grow colinear with the dominant eigenvector. It numerically infeasible to express vectors of the Krylov subspace in such a basis. For this reason, Krylov solvers usually perform a subsequent orthonormalization of the Krylov basis, which cost often defines that of the solver. For some Krylov solvers, randomization can speed up this orthonormalization process, while preserving performance and stability. In particular, we show that randomized orthogonal projections over the Krylov subspace produce quasi-optimal results.

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